# The Canonical Blow-Up of a Numerical Semigroup

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On a much deeper level, numerical semigroups induce an interesting class of one-dimensional Cohen-Macaulay local rings.

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$$\langle a_1, \ldots, a_n \rangle = \{c_1 a_1 + \cdots + c_n a_n \mid c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0}\}.$$

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$$\langle a_1, \dots, a_n \rangle = \{c_1 a_1 + \cdots + c_n a_n \mid c_1, \dots, c_n \in \mathbb{Z}_{\geq 0}\}.$$
3. Given any nonempty subset $S \subseteq \mathbb{Z}_{\geq 0}$, we write $S^* = S \setminus \{0\}$.

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Conditions (1.) and (2.) combined say that $S$ is a submonoid of $\mathbb{Z}_{\geq 0}$.

# Representations of Numerical Semigroups

**Theorem 2.7, García-Sánchez and Rosales**

Every numerical semigroup $S$ admits a unique finite minimal system of generators $S^* \setminus (S^* + S^*)$. Put another way, there exist unique integers $a_1, \ldots, a_n \in S^*$ such that $S = \{c_1 a_1 + \cdots + c_n a_n \mid c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0}\}$.

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**Lemma 2.1, García-Sánchez and Rosales**

Let $S$ be a submonoid of $\mathbb{Z}_{\geq 0}$ with $S^* \setminus (S^* + S^*) = \{a_1, \dots, a_n\}$. We have that $S$ is a numerical semigroup if and only if $\gcd(a_1, \dots, a_n) = 1$.

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Consequently, every numerical semigroup is of the form $\langle a_1, \dots, a_n \rangle$ for some unique positive integers $a_1, \dots, a_n$ such that $\gcd(a_1, \dots, a_n) = 1$. Particularly, we have that $S^* \setminus (S^* + S^*) = \{a_1, \dots, a_n\}$.

# Basic Invariants of Numerical Semigroups

Let $S$ be a numerical semigroup.

1. Considering that $\mathbb{Z}_{\geq 0} \setminus S$ is finite, we may refer to the positive integer $\mathrm{F}(S) = \max\{n \mid n \in \mathbb{Z}_{\geq 0} \setminus S\}$ as the *Frobenius number* of $S$.

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2. We define the *multiplicity* $e(S) = \min\{n \mid n \in S^*\}$.
3. We refer to the cardinality $\mu(S)$ of the unique minimal system of generators of $S$ as its *embedding dimension*.

By the Pigeonhole Principle, we must have that $\mu(S) \leq e(S)$. We say that $S$ has *maximal embedding dimension* if equality holds.

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Consider the numerical semigroup $S = \langle 3, 7, 8 \rangle$. We have that $\mathrm{F}(S) = 5$, $e(S) = 3$, and $\mu(S) = 3$, hence $S$ has maximal embedding dimension.

# Pseudo- and Difference-Frobenius Numbers

Let $S$ be a numerical semigroup. We define the *pseudo-Frobenius numbers*

$$\mathrm{PF}(S) = \{n \in \mathbb{Z}_{\geq 0} \setminus S \mid n + s \in S \text{ for all elements } s \in S^*\}.$$

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$$\text{DF}(S) = \{\text{F}(S) - x \mid x \in \text{PF}(S)\}.$$

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# Example Numerical Semigroup, Cont'd

**Example.**

Consider the numerical semigroup $S = \langle 3, 7, 8 \rangle$. We have that

$$\mathrm{PF}(S) = \{4, 5\} \text{ and } \mathrm{DF}(S) = \{0, 1\}.$$

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Particularly, $S$ is a divisive numerical semigroup.

# Symmetric Numerical Semigroups

Let $S$ be a numerical semigroup. We say that $S$ is *symmetric* if

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Consequently, we have that $S$ is symmetric if and only if $\mathrm{PF}(S) = \{\mathrm{F}(S)\}$. Particularly, divisive numerical semigroups are never symmetric.

# Divisive Numerical Semigroups Generated by Intervals

**Theorem.**

Let $S$ be a numerical semigroup generated by an interval, i.e., let

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for some integers $m > n \geq 2$.

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1. The least non-negative residue of $m-1$ modulo $n$ is not equal to 1 and either (a.) $m$ is odd or (b.) $m$ is even and $n \geq 3$.

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1. The least non-negative residue of $m - 1$ modulo $n$ is not equal to 1 and either (a.) $m$ is odd or (b.) $m$ is even and $n \geq 3$.
2. $S$ is divisive, i.e., we have that $\mathrm{F}(S) - 1 \notin S$.

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**Example.**

Consider the numerical semigroup $S = \langle 5, 6, 7, 8 \rangle$. Observe that $\mathrm{F}(S) = 9$ and $\mathrm{F}(S) - 1 = 8$ belongs to $S$, hence $S$ is not divisive.

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Even more, it can be shown that if the least non-negative residue of $m - 1$ modulo $n$ is 1, then the numerical semigroup $S = \langle m, m+1, \dots, m+n \rangle$ with $m > n \geq 2$ always admits $\mathrm{F}(S) - 1 \in S$.

# Ideals of a Numerical Semigroup

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We denote by $C = \{\mathrm{F}(S) - n \mid n \in \mathbb{Z} \setminus S\}$ the *relative canonical ideal* of $S$.

# Finitely Generated Ideals of a Numerical Semigroup

We say that a (relative) ideal $I$ of a numerical semigroup $S$ is *finitely generated* if there exist elements $x_1, \ldots, x_n \in I$ such that

$$I = \{x_1, \ldots, x_n\} + S = \{x_i + s \mid 1 \leq i \leq n \text{ and } s \in S\}.$$

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Every numerical semigroup is *Noetherian*. Particularly, every proper ideal of a numerical semigroup is finitely generated. For example, the maximal ideal $S^*$ is finitely generated by the minimal generators of $S$.

Crucially, the relative canonical ideal $C$ of $S$ is also finitely generated by $\{\mathrm{F}(S) - x \mid x \in \mathrm{PF}(S)\}$, but the proof is beyond the scope of this talk.

Let $I$ be a proper ideal of a numerical semigroup $S$ generated by $x_1 < \cdots < x_n$. We define the *blow-up numerical semigroup*

$$B_S(I) = S + \mathbb{Z}_{\geq 0}\langle x_i - x_1 \mid 1 \leq i \leq n \rangle.$$

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# Blow-Up Numerical Semigroups

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We will study the *canonical blow-up* $B_S(C)$ of a numerical semigroup. If $B_S(C)$ is symmetric, we say that $S$ has the *Gorenstein canonical blow-up* property. We will typically abbreviate this by saying that $S$ is GCB.

# Example Blow-Up Numerical Semigroup

**Example.**

Consider the numerical semigroup $S = \langle 3, 7, 8 \rangle$. Observe that $\mathrm{F}(S) = 5$, $\mathrm{PF}(S) = \{4, 5\}$, and $C = \{\mathrm{F}(S) - n \mid n \in \mathbb{Z} \setminus S\} = S \cup \{1, 4\}$.

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# An Elegant Description of the Canonical Blow-Up

**Proposition. (B-Dao)**

Let $S$ be a numerical semigroup with relative canonical ideal $C$ and difference-Frobenius numbers $\mathrm{DF}(S)$. We have that

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Consequently, every divisive numerical semigroup is GCB.

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Clearly, if $e(S) = 1$, then $S = \mathbb{Z}_{\geq 0} = B_S(C)$. Further, it is known that every numerical semigroup with $\mu(S) = 2$ is symmetric so that $B_S(C) = S$ is symmetric.

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**Example.**

Consider the numerical semigroup $S = \langle 4, 11, 13, 18 \rangle$. One can show that $\text{PF}(S) = \{7, 9, 14\}$ so that $B_S(C) = \mathbb{Z}_{\geq 0}\langle 4, 5, 7, 11, 13, 18 \rangle = \langle 4, 5, 7 \rangle$.

# A Non-GCB Numerical Semigroup of Multiplicity 4

Our previous proposition does not extend to the case that $e(S) = 4$.

**Example.**

Consider the numerical semigroup $S = \langle 4, 11, 13, 18 \rangle$. One can show that $\mathrm{PF}(S) = \{7, 9, 14\}$ so that $B_S(C) = \mathbb{Z}_{\geq 0}\langle 4, 5, 7, 11, 13, 18 \rangle = \langle 4, 5, 7 \rangle$. Observe that $B_S(C)$ is not symmetric because $\mathrm{F}(B_S(C)) = 6$ is not odd.

# Almost and Nearly Symmetric Numerical Semigroups

If $I, J \subseteq \mathbb{Z}$ are ideals of a numerical semigroup $S$, we define

$$I - J = \{n \in \mathbb{Z} \mid n + J \subseteq I\}.$$

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Observe that if $S$ is almost Gorenstein, then it is nearly Gorenstein.

# Maximal Embedding Dimension and Almost Symmetric

**Theorem. (B-Dao)**

Let $S$ be a numerical semigroup of maximal embedding dimension. We have that $S$ is almost symmetric if and only if it is nearly symmetric.

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We note that the condition that $S$ is almost symmetric if and only if it is nearly symmetric holds in general for one-dimensional Cohen-Macaulay local rings of minimal multiplicity with infinite residue field.

# A Non-GCB Almost Symmetric Numerical Semigroup

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We note that the assumption of the previous theorem that $S$ has maximal embedding dimension cannot be dropped.

**Example.**

Consider the numerical semigroup $S = \langle 4, 7, 9 \rangle$. Observe that $\operatorname{PF}(S) = \{5, 10\}$, hence we have that $B_S(C) = \mathbb{Z}_{\geq 0}\langle 4, 5, 7, 9 \rangle = \langle 4, 5, 7 \rangle$.

# A Non-GCB Almost Symmetric Numerical Semigroup

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# A Non-GCB Almost Symmetric Numerical Semigroup

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Consider the numerical semigroup $S = \langle 4, 7, 9 \rangle$. Observe that $\text{PF}(S) = \{5, 10\}$, hence we have that $B_S(C) = \mathbb{Z}_{\geq 0}\langle 4, 5, 7, 9 \rangle = \langle 4, 5, 7 \rangle$. We have seen in a previous example that $\langle 4, 5, 7 \rangle$ is not symmetric, hence $S$ is not GCB. On the other hand, we have that $C = S \cup \{5\}$, from which it follows that $S^* + C \subseteq S^*$ so that $S$ is almost symmetric.

Even in the case that $S$ has maximal embedding dimension, the converse of the previous theorem does not hold.

# A Non-Almost Symmetric GCB Numerical Semigroup

Even in the case that $S$ has maximal embedding dimension, the converse of the previous theorem does not hold.

**Example.**

Consider the numerical semigroup $S = \langle 3, 7, 8 \rangle$. We have already seen that $S$ is divisive, hence $S$ is GCB;

# A Non-Almost Symmetric GCB Numerical Semigroup

Even in the case that $S$ has maximal embedding dimension, the converse of the previous theorem does not hold.

**Example.**

Consider the numerical semigroup $S = \langle 3, 7, 8 \rangle$. We have already seen that $S$ is divisive, hence $S$ is GCB; however, we have that $C = S \cup \{1\}$, and the element $3 + 1 = 4 \in (S^* + C) \setminus S^*$ implies that $S$ is not almost symmetric.

# Questions

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# References


- D.C. Beck and H. Dao, *Canonical Blow-Up of One-Dimensional Singularities*, in progress.
- M. Delgado, P.A. García-Sánchez, and J. Morais, *numericalsgps*: a GAP System package on numerical semigroups, http://www.gap-system.org/Packages/numericalsgps.html.
- R. Fröberg, *The Frobenius number of some semigroups*, Commutative Algebra, **22**, 6021-6024, 1994.
- P.A. García-Sánchez and J.C. Rosales, *Numerical semigroups generated by intervals*, Pacific Journal of Mathematics, **191**, 1999.
- P.A. García-Sánchez and J.C. Rosales, *Numerical Semigroups*, Developments in Mathematics, **20**, Springer, 2009.

# References


J. Herzog, *Generators and relations of abelian semigroups and semigroup rings*, LSU Historical Dissertations and Theses, **1663**, 1969.

J. Herzog, T. Hibi, and D.I. Stamate, *The trace of the canonical module*, Israel Journal of Mathematics, **233**, 133-165, 2019.

J. Herzog, S. Kumashiro, and D.I. Stamate, *The tiny trace ideals of the canonical modules in Cohen-Macaulay rings of dimension one*, arXiv preprint, arXiv 2106.09404v1, 2021.

A. Moscariello and F. Strazzanti, *Nearly Gorenstein vs almost Gorenstein affine monomial curves*, Mediterranean Journal of Mathematics, **18**(127), 2021.