Research Statement

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Broadly, my mathematical research interests lie in the theory of commutative algebra. Put simply, commutative algebra is the study of basic arithmetic operations such as addition and multiplication. We refer to a set R that contains 0 as a *ring* whenever addition, subtraction, and multiplication can be performed on any pair of objects in R such that the resulting sum, difference, or product belongs to R and the familiar distributive laws of addition and multiplication hold. Given that the product of two elements r and s of R satisfies $r \cdot s = s \cdot r$, we say that R is *commutative*; if there exists a unique element 1 in R such that $1 \cdot r = r = r \cdot 1$ holds for every element r of R, we say that R is *unital*.

Even though their structure is completely determined by the operations of addition and multiplication, commutative unital rings give rise to a wealth of interesting and beautiful properties. Essential to our understanding is the introduction of additional algebraic structures such as *ideals* and *modules*. Concretely, an ideal I of a commutative ring R is a collection of ring elements such that (a.) the sum of any two elements of I yields an element of I and (b.) the product of any element of I by an arbitrary ring element yields an element of I. On the other hand, a module M over a commutative ring R is a collection of elements (not necessarily lying in R) such that any pair of elements can be added and multiplication $r \cdot m$ can be performed on any pair with r lying in R and m lying in M.

Polynomials are perhaps the most familiar class of commutative rings. Polynomials behave intuitively; however, if we restrict ourselves only to the variables $x^{s_1}, \dots, x^{s_n}$ for some positive integers $s_1, \dots, s_n$, things become much more interesting. Concretely, the polynomial $f(x) = 3x^8 - x^5 + x^2 + 1$ makes sense in the ring of polynomials with rational coefficients $\mathbb{Q}[x]$, but $f(x)$ is no longer feasible in the variables x^3 and x^7 since the only powers of x appearing in such a polynomial must be of the form $3a + 7b$ for some non-negative integers a and b. Consequently, polynomials model more sophisticated problems in additive number theory such as the *membership problem* of a *numerical semigroup*.

Even more, we may associate to any finite simple graph G on n vertices a squarefree monomial ideal $I(G)$ of the n-variate polynomial ring $k[x_1, \dots, x_n]$ over a field k. By this correspondence, we may deduce information about G via the properties of $I(G)$ and vice-versa: one can show that the ideal $I(G)$ is the *Stanley-Reisner ideal* associated to the *independence complex* of the graph G. Ultimately, these identifications forge a strong relationship — known as Stanley-Reisner theory — between the study of polynomial rings and combinatorics that has been fruitfully implemented for decades.

Broadly, the algebraic structures defined in the previous two paragraphs are *monomial algebras*. By themselves, monomial algebras constitute subtle and interesting structures whose study is challenging yet enjoyable; however, monomial algebras play an important role in bioinformatics, signal processing, and systems biology that make them invaluable in applications and mathematical modeling. Consequently, my research seeks to understand the nature of these monomial algebras.

1 Numerical Semigroups and Numerical Semigroup Rings

Univariate polynomial rings over fields enjoy well-studied and well-understood properties; however, if we restrict our attention to polynomials in $x^{a_1}, \dots, x^{a_n}$ for some indeterminate x and positive integers $a_1 < \cdots < a_n$, the situation becomes much more complicated. Central to this study, the *polynomial membership problem* seeks to determine whether a polynomial in x can be written as a polynomial in $x^{a_1}, \dots, x^{a_n}$. Concretely, observe that $x^{12} - 17x^7 + 3x^4 + 2x^3$ is a polynomial in x^3 and x^4 because

$12 = 3 \cdot 4$ and $7 = 3 + 4$ yield that $x^{12} = (x^4)^3 = (x^3)^4$ and $x^7 = x^3 \cdot x^4$; however, it is easily verified that $x^5 - x^2 + x$ cannot be written as a polynomial in x^3 and x^4 simply because of the x and x^2 terms.

Considering that multiplication of the monomials x^a and x^b corresponds to addition of the exponents a and b since $x^a \cdot x^b = x^{a+b}$, the polynomial membership problem is equivalent to the *additive membership problem* of the exponents of the monomials that generate the *monomial subring* $k[x^a, x^b]$ of $k[x]$. Based on this observation, it is fruitful to study the *numerical semigroup* S consisting of the non-negative integer powers of the monomials x^s to which we restrict our consideration.

Let k be a field. Consider the monomial subring $k[x^s \mid s \in S]$ of $k[x]$ consisting of all polynomials in the monomials x^s for some subset $S \subseteq \mathbb{Z}_{\geq 0}$ of the non-negative integers that contains 0, is closed under addition, and satisfies $\mathbb{Z}_{\geq 0} \setminus S$ is finite. We refer to the nonempty set S as a *numerical semigroup*; the commutative unital ring $k[S] = k[x^s \mid s \in S]$ is its corresponding *numerical semigroup ring.* Crucially, observe that x^a belongs to $k[S]$ if and only if the integer a belongs to S. Observe that the *Frobenius number* $\mathrm{F}(S) = \max(\mathbb{Z}_{\geq 0} \setminus S)$ is well-defined. We define the *pseudo-Frobenius numbers* of S as

$$\mathrm{PF}(S) = \{n \in \mathbb{Z}_{\geq 0} \setminus S \mid n + s \in S \text{ for all nonzero elements } s \in S\}.$$

One can establish that there exist relatively prime integers $a_1 < \cdots < a_n \in S$ such that S consists precisely of all $\mathbb{Z}_{\geq 0}$-linear combinations of these integers $a_1, \ldots, a_n$, i.e., $S = \mathbb{Z}_{\geq 0}\langle a_1, \ldots, a_n\rangle$. Even more, one may take the generators $a_1, \ldots, a_n$ to be minimal among all sets of generators of S. We refer to the minimum number of generators of S as the *embedding dimension* $\mu(S)$; the least positive integer belonging to S is its *multiplicity* $e(S)$. We note that $\mu(S) \leq e(S)$ holds by the Pigeonhole Principle.

On their own, numerical semigroups have been studied at length by Fröberg (see [Fr94]), García-Sánchez (see [AsGS, DelG-SR-P, G-SR99, G-SR09]), Herzog (see [He69, HeHiSta21, HeKumSta]), and many others because the numerical semigroup rings constitute an interesting class of one-dimensional Cohen-Macaulay rings whose properties are intimately connected with the properties of the corresponding numerical semigroup. Concretely, one can show that the embedding dimension and (Hilbert-Samuel) multiplicity of $k[S]$ and S are equal. Kunz showed that the numerical semigroup ring $k[S]$ is Gorenstein if and only if the numerical semigroup S is symmetric (see [Kun]). Even more, a numerical semigroup of embedding dimension two is Gorenstein. In a landmark result [He69, Theorem 4.2.1] of his 1969 thesis, Herzog showed that a numerical semigroup of embedding dimension three is Gorenstein if and only if it is a complete intersection. Later, Fröberg proved that the Cohen-Macaulay type of $k[S]$ is equal to the cardinality of the set of pseudo-Frobenius elements of S (see [Fr94]).

Originally motivated by an interesting new class of numerical semigroups — called *far-flung Gorenstein* — introduced by Herzog-Kumashiro-Stamate in [HeKumSta], in joint work [BeDao] with Hailong Dao, we define the *difference-Frobenius* numbers of a numerical semigroup S as

$$\mathrm{DF}(S) = \{\mathrm{F}(S) - x \mid x \in \mathrm{PF}(S)\},$$

where $\mathrm{F}(S)$ is the Frobenius number of S and $\mathrm{PF}(S)$ is the set of pseudo-Frobenius numbers of S. We say that S is *divisive* whenever 1 is a difference-Frobenius number. By definition, a divisive numerical semigroup is not symmetric. We find that a far-flung Gorenstein numerical semigroup is divisive, but the converse does not hold in general. It turns out that the divisive numerical semigroups that have certain simple or desirable properties can sometimes be described elegantly. Particularly, we determine completely those numerical semigroups generated by intervals that are divisive, and we show that a numerical semigroup of *minimal multiplicity* (i.e., *maximal embedding dimension*) is divisive if and only if its largest and second-largest generators differ by 1. We exhibit a class of "sparse" numerical semigroups with embedding dimension four that are divisive. Ultimately, we ask the following.

Question 1.1. Other than numerical semigroups of minimal multiplicity and those that are generated by intervals, for what classes of "well-behaved" numerical semigroups is it tractable to completely determine the circumstances under which such a numerical semigroup is divisive?

Let I be a subset of $\mathbb{Z}$. We say that I is an *ideal* of S if $I \supseteq S + I = \{s + i \mid s \in S \text{ and } i \in I\}$ and $s + I \subseteq S$ for some integer $s \in S$. Put another way, I must be closed under addition by elements of S and must admit a smallest element (i.e., a largest negative integer). We refer to a numerical semigroup S with maximal ideal $\mathfrak{M} = S \setminus \{0\}$ and canonical ideal $\Omega = \{-n \mid n \in \mathbb{Z} \setminus S\}$ as *nearly Gorenstein* if $\mathfrak{M} \subseteq \Omega + (S - \Omega)$ and *almost Gorenstein* if $\mathfrak{M} + \Omega = \mathfrak{M}$, where $S - \Omega = \{m \in \mathbb{Z} \mid m + \Omega \subseteq S\}$ is the ideal of differences between S and Ω. Observe that any almost Gorenstein numerical semigroup is nearly Gorenstein; the converse does not hold. Recent results of [MosStr] and [HeHiSta19, HeHiSta21] suggest an interesting connection between these classes of numerical semigroups and the trace ideal of the canonical module of the numerical semigroup ring $k[\![S]\!]$. Based on this, we define the *blow-up* of a numerical semigroup S with respect to a proper ideal $I \subseteq S$ generated by $x_1 < \cdots < x_n$ as

$$B_S(I) = S + \mathbb{Z}_{\geq 0}\langle x_i - x_1 \mid 1 \leq i \leq n\rangle.$$

We say that a numerical semigroup S has the *Gorenstein canonical blow-up* (GCB) property whenever its canonical blow-up $B_S(\Omega)$ is symmetric. Our main theorem of [BeDao] demonstrates that many interesting classes of numerical semigroups exhibit the GCB property: indeed, Arf, far-flung Gorenstein, divisive, and numerical semigroups of multiplicity at most three are GCB. Even more, if S has minimal multiplicity, then the GCB property is equivalent to the condition that S is almost Gorenstein.

Question 1.2. For what other classes of numerical semigroups is it tractable to completely determine the circumstances under which such a numerical semigroup is GCB?

Even though numerical semigroups exhibit subtle behavior, they provide interesting research questions that are well-suited for undergraduate mathematics students. Concretely, students can readily run computations and experiment to form conjectures using the [GAP System] computer software. One immediate direction regarding Questions 1.1 and 1.2 is to investigate *pinched discrete interval* numerical semigroups, which are derived from numerical semigroups generated by intervals by removing some elements. Recently, the author has classified all pinched discrete interval numerical semigroups of the form $\langle n, n+1, 2n-2, 2n-1\rangle$ as symmetric, divisive, or neither (see [Be, Chapter 3, Section 5]).

Other than their extensive importance in commutative algebra, numerical semigroups find applications in algebraic coding theory and cryptography (see [CFM], [DelFG-SL13], and [DelFG-SL14]).

2 Properties of Stable and Trace Ideals of Curve Singularities

Commutative algebraists classify the "largeness" of a commutative unital local ring R primarily through two invariants: *depth* and *Krull dimension*, denoted by $\operatorname{depth}(R)$ and $\dim(R)$, respectively. Briefly put, depth measures "homological largeness" and Krull dimension measures "topological largeness." If these two invariants coincide, i.e., $\operatorname{depth}(R) = \dim(R)$, we say that R is *Cohen-Macaulay*. Vast amounts of research have been dedicated to the study of Cohen-Macaulay rings (see [BrHe]).

Gorenstein rings form a strict subclass of Cohen-Macaulay rings that have been studied extensively for decades (see [BrHe], [Fox], [HeKun], or [LeuWie]). Crucially, a Gorenstein local ring R admits a *canonical module* ω_R that is "structurally equivalent" to R, i.e., we have that $\omega_R \cong R$ as R-modules. Even more, a canonical module of a Noetherian local ring $(R, \mathfrak{m})$ behaves well with respect to the topological process of $\mathfrak{m}$-*adic completion*. Particularly, if we denote by $\widehat{R}$ the $\mathfrak{m}$-adic completion of R, we have that $\omega_{\widehat{R}} \cong \widehat{\omega_R}$ is a canonical module of the Noetherian local ring $(\widehat{R}, \mathfrak{m}\widehat{R})$. We say that R is *analytically unramified* if $\widehat{R}$ is *reduced*, i.e., every power of a nonzero element of $\widehat{R}$ is nonzero.

We assume throughout this section that $(R, \mathfrak{m}, k)$ is an analytically unramified one-dimensional Cohen-Macaulay local ring with infinite residue field k, total ring of fractions $Q(R)$, and integral closure $\overline{R}$. Under these conditions, it is known that R possesses a canonical ideal $\omega_R \subseteq R$ (see [HeKun]) and every $\mathfrak{m}$-primary ideal of R has a principal reduction (cf [HuSw, Proposition 8.3.7]). Based on recent work of Herzog, Stamate, et al. (see [HeHiSta19, HeHiSta21, HeKumSta]), the author has sought in joint work with Hailong Dao to understand the role that the canonical ideal ω_R plays in determining the structure of R. Generally, our work aims to answer questions about when the canonical blow-up

$$B(\omega_R) = \bigcup_{n \geq 0} (\omega_R^n : \omega_R^n) = \{\alpha \in Q(R) \mid \alpha \omega_R^n \subseteq \omega_R^n\}$$

is Gorenstein, almost Gorenstein, or nearly Gorenstein, or satisfies that $B(\omega_R) = \overline{R}$ (see [BeDao]).

Using the properties of stable ideal theory outlined in the seminal work of Lipman [Lip] in conjunction with techniques related to trace ideals described in recent work of Dao [Dao21], Dao-Lindo [DaoLin], Dao-Maitra-Sridhar [DaoMaSr], and Kobayashi-Takahashi [KoTa], we have determined that the properties of $B(\omega_R)$ are intimately related to the behavior of sufficiently large powers of ω_R and their trace ideals. Particularly, the Gorensteinness of $B(\omega_R)$ is equivalent to any of the following.

(1.) There exists an $n \gg 0$ such that $\omega_R^n \cong (\omega_R^n)^\vee$, where $-^\vee$ denotes the canonical dual of $-$.

(2.) There exists an $n \gg 0$ such that ω_R^n is self-dual.

(3.) There exists an $n \gg 0$ such that $\operatorname{tr}(\omega_R^n)$ is stable.

Further, we show that any Arf ring has the GCB property. Using the dual of the canonical blow-up $b(\omega_R) = (R : B(\omega_R))$, we exhibit equivalent properties for R to be almost Gorenstein. For instance, if $\mathfrak{m}$ is ω_R-Ulrich or $\mathfrak{m} \subseteq b(\omega_R)$, then R is almost Gorenstein; the converses hold, as well.

Extensive effort has been made jointly by Dao and the author toward the case that $R_S = k[\![S]\!]$ is the numerical semigroup ring corresponding to the numerical semigroup S. Observe that a numerical semigroup ring is a complete one-dimensional Noetherian local domain and hence Cohen-Macaulay. Further, by a result of Nagata in [Na], the integral closure of a numerical semigroup ring is module-finite, hence the integral closure of R_S is a module-finite birational extension of R_S. Using the methods described in the last two paragraphs of the previous section, there is an elegant description of the canonical blow-up of R_S as an R_S-algebra whose generators are simply the difference-Frobenius numbers. Consequently, the properties of the canonical blow-up of R_S are intimately related to the properties of the canonical blow-up of S. We show that the divisive numerical semigroup rings are a natural generalization of the far-flung Gorenstein numerical semigroup rings of Herzog-Kumashiro-Stamate.

3 Additive Number Theory, Sumsets, and Monomial Subrings

Consider a nonempty (not necessarily finite) subset $S \subseteq \mathbb{Z}_{\geq 0}$. We say that S is *Sidon* if for every pair of non-negative integers $i \leq j$, the sum $s_i + s_j$ of the elements $s_i, s_j \in S$ is unique. Put another way, there do not exist distinct pairs of integers $i \leq j$ and $i' \leq j'$ such that $s_i + s_j = s_{i'} + s_{j'}$ for some elements $s_i, s_j, s_{i'}, s_{j'} \in S$. Originally introduced by Simon Sidon in his study of Fourier series, Sidon sets attracted significant interest in the field of additive number theory after a result of Erdös and Turán showed that for every real number $x > 0$, the number of elements of a Sidon set that do not exceed x is at most $\sqrt[4]{x} + O(\sqrt[4]{x})$ (see [ErdTu]). Even now, it remains an open problem to determine the maximum number of elements not exceeding a real number $x > 0$ that a Sidon set can contain.

One may consider the question of Sidon as to how "dense" a Sidon set can be if its largest element does not exceed some real number $x > 0$. Conversely, given a positive integer m, one may ask the question of how "sparse" a set can be such that the n-fold sum of its elements achieves a maximum value of m. Colloquially, this is known as the Postage Stamp Problem due to the following interpretation: if an envelope affords enough space for n stamps and there are k denominations of stamps available, what is the maximum cost of postage m such that any letter of cost $0, 1, \dots, m$ can be mailed?

Given any nonempty subset A of non-negative integers, we define the n-fold sum of A as

$$nA = \underbrace{A + \cdots + A}_{n \text{ summands}} = \{a_1 + \cdots + a_n \mid a_1, \ldots, a_n \in A\},$$

and we denote by $[m] = \{0, 1, \ldots, m\}$ the discrete interval $[0, m] \cap \mathbb{Z}$. Considered the first to state the Postage Stamp Problem, Rohrbach defined the invariants $m(n, A) = \max\{m : \{0, 1, \ldots, m\} \subseteq nA\}$ and $m(n, k) = \max\{m(n, A) : |A| = k\}$ in his seminal 1937 paper [Roh]. Even though it is relatively simple to state, it has been shown that the computational complexity of the problem is exponential in both n and k, and there remain many open questions regarding $m(n, k)$ (see [AlBa]).

One particular case of the Postage Stamp Problem centers around the notion of a complete double. Consider any nonempty subset $A \subseteq [a]$. We say that A is a *complete double* of $[a]$ if $2A = [2a]$. Clearly, every discrete interval $[a]$ is a complete double of itself, hence one might ask for the least cardinality $\mu(a) = \min\{|A| : 2A = [2a]\}$ of A such that A is a complete double of a. Based on the discussion to a post by Hailong Dao on MathOverflow (see [Dao19]), the author has established mild bounds for $\mu(a)$ in [Be, Chapter 5, Section 2]. Further, in the same section, the author has defined the *regularity* $\operatorname{reg}(A) = \inf\{n \mid nA = [na]\}$ of A and exhibited necessary and sufficient conditions to guarantee that $\operatorname{reg}(A)$ is finite. Even still, questions remain regarding an explicit connection between $|A|$ and $\operatorname{reg}(A)$.

Question 3.1. Is there an elegant relationship between $|A|$ and $\operatorname{reg}(A)$?

Question 3.2. What are the bounds on $|A|$ such that $\operatorname{reg}(A) = 3$, i.e., A is a complete triple of $[a]$?

Question 3.3. Is there a sharper bound between $\operatorname{reg}(A)$ and $a - 2 = \max\{\operatorname{reg}(A) : \operatorname{reg}(A) < \infty\}$?

Curiously, the theory of complete doubles has interesting and important applications in homological algebra. Given any subset $\mathcal{A} \subseteq [a]$ that contains $\{0, 1, a-1, a\}$, we may define the monomial subring

$$k[x, y]^{(\mathcal{A})} = k[x^i y^{a-i} \mid i \in \mathcal{A}] \subseteq k[x, y].$$

Crucially, the homological properties of $k[x, y]^{(\mathcal{A})}$ are intimately connected with the number-theoretic properties of $\mathcal{A}$. We have computed in [Be, Chapter 5, Section 3] the Hilbert series, (Hilbert-Samuel) multiplicity, and (Castelnuovo-Mumford) regularity of $k[x, y]^{(\mathcal{A})}$ in terms of the invariants of $\mathcal{A}$; thus, we have reduced sophisticated and important algebraic questions to a more tractable setting of additive number theory in which the techniques involved in many of the underlying proofs and computations are viable to any undergraduate mathematics student with a basic understanding of calculus.

Conversely, *planar additive bases* are closely related to the study of complete doubles (see [KKR]) and enjoy applications in signal processing and active imaging (see [HoKa] and [KoKa]). Consequently, we are afforded algebraic techniques to solve an inherently number-theoretic optimization problem.

4 Combinatorial Commutative Algebra and Stanley-Reisner Theory

We say that a pair $G = (V, E)$ consisting of sets V of *vertices* and $E \subseteq V \times V$ of *edges* is a *finite simple graph* whenever V consists of finitely many elements and E does not contain any *loops* or *multiple edges*. Loops are pairs of vertices of the form (v, v), and an edge (v, w) is called a multiple edge if (w, v) is also an edge. Conventionally, we assume that the n vertices of G are the positive integers 1 through n, i.e., $V = \{1, 2, \ldots, n\}$. We say that a nonempty collection of vertices V' constitutes an *independent vertex set* if for any two vertices $i, j \in V'$, we have that (i, j) is not an edge of G.

Generalizing the notion of a finite simple graph is that of a *simplicial complex* Δ. We say that Δ is a simplicial complex on the vertex set $[n] = \{1, 2, \ldots, n\}$ if Δ is a nonempty subset of $2^{[n]}$ such that for every pair of subsets $\sigma, \tau \in 2^{[n]}$ with $\tau \subseteq \sigma$, if σ belongs to Δ, then τ belongs to Δ. Put another way, Δ must be closed with respect to the operation of taking subsets. We point out that some familiar geometric objects — such as line segments, triangles, and tetrahedra — are simplicial complexes.

Every finite simple graph on n vertices gives rise to a quotient of the polynomial ring in n variables. Explicitly, for a field k, a finite simple graph $G = (V, E)$ on n vertices can be related to the quotient ring $k(G) = k[x_1, \dots, x_n]/I(G)$ by the squarefree monomial ideal $I(G) = (x_i x_j \mid (i, j) \in E)$. We refer to $I(G)$ as the *edge ideal* of G and to $k(G)$ as the *edge ring* of G. Likewise, every simplicial complex on n vertices gives rise to a quotient of the polynomial ring in n variables. For simplicity, we use the same residue field. Explicitly, we define the *Stanley-Reisner ring* $k[\Delta] = k[x_1, \dots, x_n]/I_\Delta$, where

$$I_\Delta = (x_{i_1} x_{i_2} \cdots x_{i_k} \mid \{i_1, i_2, \dots, i_k\} \subseteq 2^{[n]} \setminus \Delta)$$

is the *Stanley-Reisner ideal* of $k[x_1, \dots, x_n]$ generated by the monomials corresponding to subsets of $2^{[n]}$ that do not belong to Δ. Often, the elements of Δ are referred to as *faces*, hence I_Δ is generated by monomials corresponding to non-faces of Δ. We do not assume that all of the integers of $[n]$ correspond to vertices of Δ, hence it is possible that x_i belongs to I_Δ for some integer $1 \leq i \leq n$ so that $k[\Delta]$ is a quotient of the polynomial ring in fewer than n elements. We recall that the set Δ_G consisting of the independent vertex sets of G is a simplicial complex, eponymously called the *independence complex* of G. One can show that the edge ideal of G and the Stanley-Reisner ideal of Δ_G are equal, hence we have that $k(G) = k[\Delta_G]$ (see [MooRogSa-Wa, Theorem 4.4.9]). Consequently, the so-called Stanley-Reisner theory can be employed to understand properties of the edge ring $k[G]$ and vice-versa.

Recently, the author introduced in joint work [BeDey] with Souvik Dey the following pair of new invariants $\operatorname{ms}(R)$ and $\operatorname{cs}(R)$ of a Noetherian (standard graded) local ring $(R, \mathfrak{m}, k)$.

$$\begin{aligned}\operatorname{cs}(R) &= \min\{\dim_k(I/\mathfrak{m}I) \mid I \subseteq \mathfrak{m} \text{ is a (homogeneous) ideal of } R \text{ such that } \mathfrak{m}^2 = I^2\}\\ \operatorname{ms}(R) &= \min\{\dim_k(I/\mathfrak{m}I) \mid I \subseteq \mathfrak{m} \text{ is a (homogeneous) ideal of } R \text{ such that } \mathfrak{m}^2 \subseteq I\}.\end{aligned}$$

Particularly, we sought to understand these invariants for the edge ring $k(G)$ of a finite simple graph G and a fixed field k. Conventionally, we write $\operatorname{ms}(G)$ and $\operatorname{cs}(G)$ for $\operatorname{ms}(k(G))$ and $\operatorname{cs}(k(G))$, respectively. Even though we can compute $\operatorname{ms}(R)$ and $\operatorname{cs}(R)$ in some cases, it turns out that these invariants behave in a surprisingly subtle manner in the setting of edge rings of finite simple graphs. We proved that

$$\operatorname{ms}(K_n) = 1 \text{ and } \operatorname{cs}(K_n) = \left\lceil \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \right\rceil$$

for the complete graph K_n on n vertices and that $\operatorname{ms}(S_n) = n - 1$ and $\operatorname{cs}(S_n) = n$ for the star graph S_n on n vertices. Consequently, we believe that $\operatorname{ms}(G)$ and $\operatorname{cs}(G)$ are determined in some way by the "connectivity" and number of edges of G; the exact relationship is as yet undetermined.

Crucially, we have illustrated that $\operatorname{ms}(G)$ does not change with respect to adding a cone vertex to G (i.e., taking the graph join of G with a new vertex), hence it suffices to study $\operatorname{ms}(G)$ in the case that G has diameter two. We have found along these lines that if G admits a "sufficiently connected" edge cover by ℓ edges, then $\operatorname{ms}(G) \leq \ell$; however, it is unclear if every finite simple graph of diameter two admits such an edge cover. Our future work will therefore concern the following two questions.

Question 4.1. Can we compute $\operatorname{ms}(G)$ and $\operatorname{cs}(G)$ for many classes of familiar graphs? Bounds have been given for (a.) the path graph P_n, (b.) the cycle graph C_n, and (c.) the wheel graph W_n, but in each case, the exact value of one (or both) of $\operatorname{ms}(G)$ or $\operatorname{cs}(G)$ has not been found.

Question 4.2. If G is a finite simple graph of diameter two, then must it admit an edge cover that is K_ℓ-connected for some integer $\ell \geq 1$ (see [BeDey, Definition 6.27])? In particular, can any edge cover of G be "amended" to an edge cover that is K_ℓ-connected? By what process, if possible?

One other interesting graphical invariant arises in the context of *Gaussian graphical models* that enjoy applications in computational biology and sociology (see [XLV] and [HL], respectively). We will assume toward this end that $\mathbf{X} = (X_1, \dots, X_m)$ is an m-dimensional random vector with multivariate normal (or Gaussian) distribution $X \sim \mathcal{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ is an $m \times m$ positive-semidefinite *covariance matrix*. Consider the finite simple graph G with vertices $[m]$ and edges $\{i, j\}$ if and only if the random variables X_i and X_j are conditionally dependent given all other random variables (see [Uhl, Corollary 2.2]). By the paragraph following [GroSul, Problem 1.2], the *maximum likelihood threshold* is given by

$$\text{mlt}(G) = \min\{\#\text{i.i.d. samples} \mid \boldsymbol{\Sigma} \text{ exists with probability one}\}.$$

By [BeDey, Proposition 7.2], the author has demonstrated that $\text{ms}(\mathbb{R}(G)) = \text{mlt}(\overline{G})$ if

(1.) $\overline{G}$ is chordal;

(2.) $\overline{G}$ is complete;

(3.) G is complete; or

(4.) $\overline{G}$ has no induced cycles (i.e., $\overline{G}$ is a tree).

Question 4.3. Let G be the finite simple graph corresponding to an m-dimensional Gaussian random vector. Let $I(G)$ be the edge ideal of G in $\mathbb{R}[x_1, \dots, x_m]$. Does it hold that $\text{ms}(\mathbb{R}(G)) = \text{mlt}(\overline{G})$?

Graph theory enjoys delightful applications in biology, chemistry, and computer science that attract students who are interested in applied mathematics. Likewise, questions such as those mentioned above entice students who aspire to a career in pure mathematics by interpolating techniques from discrete mathematics, linear algebra, and ring theory. Graph theory also affords the use of diagrams in tandem with computer software to carry out experiments that allow students to form and test conjectures; as such, Questions 4.1 and 4.2 provide excellent opportunities for graduate and undergraduate research.

References

[AlBa] R. Alter and J.A. Barnett, *A postage stamp problem*, The American Mathematical Monthly, **87**(3), 206-210, 1980.

[AsGS] A. Assi and P.A. García-Sánchez, *Numerical Semigroups and Applications*, hal-01085760, 2014.

[BaFr] V. Barucci and R. Fröberg, *One-Dimensional Almost Gorenstein Rings*, Journal of Algebra, **188**, 418-442, 1997.

[Be] D.C. Beck, *On a generalization of two-dimensional Veronese subrings*, in progress.

[BeDao] D.C. Beck and H. Dao, *Canonical blow-up of one-dimensional singularities*, in progress.

[BeDey] D.C. Beck and S. Dey, *Some new invariants of Noetherian local rings related to square of the maximal ideal*, arXiv preprint: 2205.01658, 2022.

[BlSi] G. Blekherman and R. Sinn, *Maximum likelihood threshold and generic completion rank of graphs*, Discrete & Computational Geometry, **61**, 303-324, 2019.

[BrHe] W. Bruns and J.R. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics, **39**, Cambridge University Press, 1993.

[CFM] A. Campillo, J.I. Farrán, and C. Munuera, *On the parameters of algebraic geometry codes related to Arf semigroups*, IEEE Transactions of Information Theory, **46**, 2634-2638, 2000.

[Dao19] H. Dao, *Size of sets with complete double*, MathOverflow.net, 2019.

[Dao21] H. Dao, *Reflexive modules, self-dual modules, and Arf rings*, preprint, arXiv:2105.12240v1, 2021.

[DaoHuSch] H. Dao, C. Huneke, and J. Schweig, *Bounds on the regularity and projective dimension of ideals associated to graphs*, Journal of Algebraic Combinatorics, **38**, 2011.

[DaoLin] H. Dao, H. Lindo, *Stable trace ideals and applications*, arXiv preprint, arXiv:2106.07064v1, 2021.

[DaoMaSr] H. Dao, S. Maitra, P. Sridhar, *On reflexive and I-Ulrich modules over curve singularities*, arXiv preprint, arXiv:2101.02641v5, 2021.

[DelG-SR-P] M. Delgado, P.A. García-Sánchez, and A.M. Robles-Pérez, *Numerical semigroups with a given set of pseudo-Frobenius numbers*, LMS J. Comput. Math., **19**(1), 186-205, 2016.

[DelFG-SL13] M. Delgado, J.I. Farrán, P.A. García-Sánchez, and D. Llena, *On the generalized Feng-Rao numbers of numerical semigroups generated by intervals*, Mathematics of Computation, **82**(283), 1813-1836, 2013.

[DelFG-SL14] M. Delgado, J.I. Farrán, P.A. García-Sánchez, and D. Llena, *On the weight hierarchy of codes coming from semigroups with two generators*, IEEE Transactions on Information Theory, **60**(1), 282-295, 2014.

[EisHuUl] D. Eisenbud, C. Huneke, and B. Ulrich, *The regularity of Tor and graded Betti numbers*, American Journal of Mathematics, **128**, 573-605, Johns Hopkins University Press, 2006.

[ErdTu] P. Erdös and P. Turán, *On a problem of Sidon in additive number theory, and on some related problems*, Journal of the London Mathematical Society, **16**, 212-215, 1941.

[Fox] H-B. Foxby, *Gorenstein modules and related modules*, Mathematica Scandinavica, **31**(2), 267-284, 1972.

[Fr90] R. Fröberg, *On Stanley-Reisner rings*, Banach Center Publications, **26**, 57-70, 1990.

[Fr94] R. Fröberg, *The Frobenius number of some semigroups*, Commutative Algebra, **22**, 6021-6024, 1994.

[GAP System] The GAP Group, *GAP: Groups, Algorithms, and Programming, Version 4.11.1*, https://www.gap-system.org, 2021.

[Gi] R. Gilmer, *Commutative Semigroup Rings*, Chicago Lectures in Mathematics, **16**, The University of Chicago Press, 1984.

[GroSul] E. Gross and S. Sullivant, *The maximum likelihood threshold of a graph*, Bernoulli, **24**(1), 386-407, 2018.

[G-SR99] P.A. García-Sánchez and J.C. Rosales, *Numerical semigroups generated by intervals*, Pacific Journal of Mathematics, **191**, 1999.

[G-SR09] P.A. García-Sánchez and J.C. Rosales, *Numerical Semigroups*, Developments in Mathematics, **20**, Springer, 2009.

[He69] J.R. Herzog, *Generators and relations of abelian semigroups and semigroup rings*, LSU Historical Dissertations and Theses, **1663**, 1969.

[HeKun] J.R. Herzog and E. Kunz, *Der kanonische Modul eines Cohen-Macaulay Rings*, Springer Lecture Notes in Mathematics, **238**, 1971.

[HeHiSta19] J. Herzog, T. Hibi, and D.I. Stamate, *The trace of the canonical module*, Israel Journal of Mathematics, **233**, 133-165, 2019.

[HeHiSta21] J. Herzog, T. Hibi, and D.I. Stamate, *Canonical trace ideal and residue for numerical semigroup rings*, Semigroup Forum, **103**, 550-566, 2021.

[HeKumSta] J. Herzog, S. Kumashiro, and D.I. Stamate, *The tiny trace ideals of the canonical modules in Cohen-Macaulay rings of dimension one*, preprint, 2021.

[HL] S. Høsgaard and S.L. Lauritzen, *Graphical Gaussian models with edge and vertex symmetries*, Journal of the Royal Statistical Society, Series B (Statistical Methodology), **70**(5), 1005-1027, 2008.

[HoKa] R.T. Hoctor and S. Kassam, *Unifying role of the coarray in aperture synthesis for coherent and incoherent imaging*, Proceedings of the IEEE, **78**(4), 735-752, 1990.

[HuSw] C. Huneke and I. Swanson, *Integral closure of ideals, rings and modules*, London Mathematical Society Lecture Note Series 336.

[KKR] J. Kohonen, V. Koivunen, and R. Rajamäki, *Planar additive bases for rectangles*, arXiv:1711.08812v1, 2017.

[KoKa] R.J. Kozick and S.A. Kassam, *Linear imaging with sensor arrays on convex polygonal boundaries*, IEEE Transactions on Systems, Man, and Cybernetics, **21**(5), 1991.

[KoTa] T. Kobayashi and R. Takahashi, *Rings whose ideals are isomorphic to trace ideals*, Mathematische Nachrichten, **292**(10), 2252-2261, 2019.

[Kun] E. Kunz, *The value-semigroup of a one dimensional Gorenstein ring*, Proceedings of the American Mathematical Society, **25**. 748–751, 1970.

[LeuWie] G. Leuschke and R. Wiegand, *Cohen-Macaulay Representations*, Mathematical Surveys and Monographs, American Mathematial Society, **181**, Providence, RI, 2012.

[Lip] J. Lipman, *Stable ideals and Arf rings*, American Journal of Mathematics, **93**, 649-685, 1971.

[MooRogSa-Wa] W.F. Moore, M. Rogers, and K. Sather-Wagstaff, *Monomial Ideals and Their Decompositions*, Springer, 2018.

[MosStr] A. Moscariello and F. Strazzanti, *Nearly Gorenstein vs almost Gorenstein affine monomial curves*, Mediterranean Journal of Mathematics, **18**(127), 2021

[Na] M. Nagata, *On the structure of complete local rings*, Nagoya Mathematical Journal, **1**, 63-70, 1950.

[Roh] H. Rohrbach, *Ein Beitrag zur additiven Zahlentheorie*, Mathematische Zeitschrift, **42**, 1-30, 1937.

[Uhl] C. Uhler, *Gaussian Graphical Models: An Algebraic and Geometric Perspective*, arXiv : 1707.04345v1, 2017.

[W] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Englewood Cliffs, NJ, 2000.

[XLV] Y. Xie, Y. Liu, and W. Valdar, *Joint Estimation of Multiple Dependent Gaussian Graphical Models with Applications to Mouse Genomics*, Biometrika, **103**(3), 2016.