Canonical Blow-Up of One-Dimensional Singularities

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We say that a Cohen-Macaulay local ring R is *Gorenstein* if $\operatorname{type}(R) = 1$.

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For an R-module M, we will denote by $M^* = \operatorname{Hom}_R(M, R)$ the *dual* of M and by $M^\vee = \operatorname{Hom}_R(M, \omega_R)$ the *canonical dual* of M.

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Let $\overline{R}$ denote the integral closure of R, i.e.,

$$\overline{R} = \{\alpha \in Q(R) \mid \alpha^n + r_1\alpha^{n-1} + \cdots + r_n = 0 \text{ for some } r_1, \ldots, r_n \in R\}.$$

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We define the *conductor* $(R : \overline{R}) = \{\alpha \in Q(R) \mid \alpha\overline{R} \subseteq R\}$ of R. Observe that $(R : \overline{R})$ is an ideal of both R and $Q(R)$.

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Conversely, if R is a one-dimensional Noetherian ring that admits a canonical ideal ω_R, then R is generically Gorenstein.

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Particularly, an analytically unramified ring admits a canonical ideal.

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Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring. We define the *Hilbert-Samuel multiplicity* of a finitely generated R-module M with respect to an $\mathfrak{m}$-primary ideal I of R as

$$e_R(I, M) = \lim_{n \to \infty} \frac{(d-1)!}{n^{d-1}} \ell_R\left(\frac{I^n M}{I^{n+1} M}\right).$$

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Preliminary Observations

We assume throughout the rest of this talk that $(R, \mathfrak{m}, k)$ is an analytically unramified one-dimensional Cohen-Macaulay local ring with infinite residue field k, total ring of fractions $Q(R)$, integral closure $\overline{R}$, and conductor $(R : \overline{R}) = \{\alpha \in Q(R) \mid \alpha\overline{R} \subseteq R\}$.

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1. R admits a canonical ideal ω_R, and
2. every $\mathfrak{m}$-primary ideal of R has a principal reduction.

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We assume throughout the rest of this talk that $(R, \mathfrak{m}, k)$ is an analytically unramified one-dimensional Cohen-Macaulay local ring with infinite residue field k, total ring of fractions $Q(R)$, integral closure $\overline{R}$, and conductor $(R : \overline{R}) = \{\alpha \in Q(R) \mid \alpha \overline{R} \subseteq R\}$. Under these assumptions,

1. R admits a canonical ideal ω_R, and
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Considering that ω_R is a regular ideal of R, we find that

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Generally, every regular ideal of R has finite colength, hence every regular ideal of R is $\mathfrak{m}$-primary, and we find that

4. every regular ideal of R has a principal reduction by a regular element.

The Blow-Up Ring of R with Respect to an Ideal I

Let I be an ideal of R. We define the *blow-up*

$$B(I) = \bigcup_{n \geq 0} (I^n : I^n) = \{\alpha \in Q(R) \mid \alpha I^n \subseteq I^n\}$$

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We refer to $B(\omega_R)$ as the *canonical blow-up* of R.

Stable Ideals

Let I be a regular ideal of R. We say that I is *stable* if any of the following equivalent conditions hold (cf. [5, Definition 1.1 and Lemma 1.3]).

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Particularly, if I is stable, then $B(I) = \frac{I}{x}$ is finitely generated as an R-module, hence $b(I)$ is finitely generated as an R-module.

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Lemma (cf. [2, Lemma 2.6]).

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If I is a regular ideal of R, then $b(I) = \operatorname{tr}(B(I))$ (cf. [2, Remark 2.3]).

Proposition (cf. [2, Proposition 2.8]).

Let I be a regular ideal of R. The following properties hold.

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The Blow-Up, the Stable Class, and (Canonical) Duality

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Let I be a regular ideal of R. The following properties hold.

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Proof.

Property (1.) holds by [5, Lemma 1.3] and [2, Lemma 2.6]. Observe that $B(I) = R[I^n/x^n]$, hence the canonical map $R \to B(I)$ is a local ring homomorphism of one-dimensional Cohen-Macaulay local rings, and $B(I)$ admits a canonical module $B(I)^\vee$; this shows property (2.).

Proof.

Last, we will establish that $\mathfrak{s}_{\omega_R} \cong \operatorname{tr}(\mathfrak{s}_{\omega_R})^* \cong (R : b(\omega_R))$. We have that

$$\mathfrak{s}_{\omega_R}^* = \operatorname{Hom}_R(B(\omega_R), R) \cong b(\omega_R) = \operatorname{tr}(B(\omega_R)) = \operatorname{tr}(\mathfrak{s}_{\omega_R}).$$

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By [1, Corollary 4.29], the canonical stable class $\mathfrak{s}_{\omega_R}$ is reflexive, hence we conclude that $\mathfrak{s}_{\omega_R} = \mathfrak{s}_{\omega_R}^{**} \cong \operatorname{tr}(\mathfrak{s}_{\omega_R})^* = b(\omega_R)^* \cong (R : b(\omega_R))$. QED.

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If R is Arf, then R is GCB.

Nearly Gorenstein and Almost Gorenstein Rings

Let $(S, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring such that $\overline{S}$ is S-module-finite and S admits a canonical module $S \subseteq C \subseteq \overline{S}$. We say that S is *almost Gorenstein* if any of the following equivalent conditions holds.

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Observe that if $\mathfrak{m}C = \mathfrak{m}$, then $\mathfrak{m} = \operatorname{tr}(\mathfrak{m}) = \operatorname{tr}(\mathfrak{m}C) = \mathfrak{m} \operatorname{tr}(C) \subseteq \operatorname{tr}(C)$.

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Proposition (cf. [2, Proposition 2.18]).

Suppose that $\overline{R}$ is finitely generated as an R-module, e.g., R is reduced. If R has minimal multiplicity, then the following conditions are equivalent.

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Proof.

Conditions (1.) and (2.) are equivalent by [6, Theorem 6.6] or [6, Corollary 8.4], so it suffices to show that (2.) implies that R is GCB.

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Proof.

Conditions (1.) and (2.) are equivalent by [6, Theorem 6.6] or [6, Corollary 8.4], so it suffices to show that (2.) implies that R is GCB. If R is Gorenstein, then $B(\omega_R) = R$ is Gorenstein. Consequently, we may assume that R is almost Gorenstein but not Gorenstein.

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By [4, Exercise 4.6.14], if R has minimal multiplicity, then $\mathfrak{m}$ is stable. If R is nearly Gorenstein, then $\text{tr}(\mathfrak{s}_{\omega_R}) = \text{tr}(B(\omega_R)) \supseteq \mathfrak{m}$. We claim that $\text{tr}(\mathfrak{s}_{\omega_R}) = \mathfrak{m}$, from which it follows that $\text{tr}(\mathfrak{s}_{\omega_R})$ is stable, hence $B(\omega_R)$ is Gorenstein.

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Connections to Numerical Semigroups

Let $a_1, \dots, a_n$ be pairwise relatively prime positive integers. Let $S = \mathbb{Z}_{\geq 0}\langle a_1, \dots, a_n \rangle$ be numerical semigroup generated by $a_1, \dots, a_n$. By Bézout's Theorem, we have that $\mathbb{Z}_{\geq 0} \setminus S$ is finite, hence we may consider the largest positive integer not contained in S.

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$$\mathrm{PF}(S) = \{n \in \mathbb{Z}_{\geq 0} \setminus S \mid n + s \in S \text{ for all elements } s \in S \setminus \{0\}\}.$$

Proposition (cf. [2, Proposition 5.3]).

Let K be an infinite field. Let $R = K[\![t^s \mid s \in S]\!]$ be the numerical semigroup ring associated to the numerical semigroup S.

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$$B(\omega_R) = R[\![t^{\mathrm{F}(S)-x} \mid x \in \mathrm{PF}(S)]\!].$$

We say that S is *divisive* if $\mathrm{F}(S) - 1 \in \mathrm{PF}(S)$ (i.e., $B(\omega_R)$ is regular).

Familiar Classes of GCB Rings

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6. R is a numerical semigroup ring of multiplicity ≤ 3.

Questions

Acknowledgments

I express my gratitude to my academic advisor and co-author Hailong Dao for his insight, expertise, and guidance in collaboration on this paper. I thank Souvik Dey for his unfailing advice in our countless conversations. Last, I would like to thank the creators of the GAP System, which I have used extensively for computations with numerical semigroups.

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