Double Integration over Rectangles

- Recall from Calculus I that we define the **definite integral** of a function $f(x)$ on an interval $[a, b]$ to be the limit

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- Like one would reasonably hope, the idea of the definite integral of $f(x)$ over a closed interval $[a, b]$ generalizes nicely to the integral of a function $f(x, y)$ over $[a, b] \times [c, d]$, i.e., a rectangle.

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- Given a function $f(x, y)$ such that the quantity

$$L = \lim_{||\mathcal{P}|| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}) \Delta x \Delta y$$

exists for all possible partitions P_{ij} of the rectangle $\mathcal{R} = [a, b] \times [c, d]$, we say that $f(x, y)$ is (Riemann) **integrable** with double integral

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- We interpret the quantity L as the *signed* volume of the region enclosed by the graph of $f(x,y)$ and the rectangle $\mathcal{R}$ in the xy-plane.

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- **Fubini's Theorem** allows use to practically compute double integrals.

Properties of Double Integrals over Rectangles

Fubini's Theorem

Given a continuous function $f(x, y)$ over a rectangle $\mathcal{R} = [a, b] \times [c, d]$, the following quantities are equal.

1. $$\iint_{\mathcal{R}} f(x, y)\, dA$$
2. $$\int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x, y)\, dy \right) dx$$
3. $$\int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x, y)\, dx \right) dy$$

True (a.) or False (b.)

The signed volume of the region bounded by the function $f(x, y) = \sin x$ and the rectangle $\mathcal{R} = [-1, 1] \times [0, 1]$ is zero.

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(a.) True. We have that

$$\iint_{\mathcal{R}} f(x, y)\, dA = \int_0^1 \int_{-1}^1 \sin x \, dx \, dy = \int_0^1 \left[-\cos x \right]_{-1}^1 dy = \int_0^1 0 \, dy$$

since $\cos x$ is an even function, i.e., $\cos(-1) = \cos(1)$.

Double Integration over General Regions

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- If $\mathcal{D}$ is vertically simple, i.e., $a \leq x \leq b$ and $f_1(x) \leq y \leq f_2(x)$,

$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx.$$

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- If $\mathcal{D}$ is horizontally simple, i.e., $c \leq y \leq d$ and $g_1(y) \leq x \leq g_2(y)$,

$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy.$$

Choosing the Best Orientation

Compute the double integral $\iint_{\mathcal{D}} \frac{\sin x}{x}\, dA$ on the closed domain $\mathcal{D} = \{(x, y) \,|\, y \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$.

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We note that $\mathcal{D}$ is described as horizontally simple, hence we have that

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But this function has no elementary antiderivative; however, if we consider $\mathcal{D}$ as defined by $0 \leq x \leq 1$ and $0 \leq y \leq x$ (vertically simple), we have that

$$\iint_{\mathcal{D}} \frac{\sin x}{x}\, dA = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx = \int_0^1 \sin x\, dx = 1 - \cos(1).$$

Reversing the Best Orientation

Compute the double integral $\iint_{\mathcal{D}} xe^{y^4}\, dA$ on the closed domain $\mathcal{D} = \{(x, y)\,|\, 0 \leq x \leq 1 \text{ and } x^{2/3} \leq y \leq 1\}$.

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But the function e^{y^4} has no elementary antiderivative; however, if we consider $\mathcal{D}$ as defined by $0 \leq x \leq y^{3/2}$ and $0 \leq y \leq 1$, we have that

$$\iint_{\mathcal{D}} xe^{y^4}\,dA = \int_0^1 \int_0^{y^{3/2}} xe^{y^4}\,dx\,dy = \frac{1}{2}\int_0^1 y^3 e^{y^4}\,dy = \frac{e-1}{8}.$$