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- Observe that a vector-valued function is a function $\mathbb{R} \to \mathbb{R}^3$, hence its domain is a subset of the real numbers. Particularly, the domain of $\mathbf{r}(t)$ is the intersection of the domains of $x(t)$, $y(t)$, and $z(t)$.

True (a.) or False (b.)

The domain of $\mathbf{r}(t) = \langle \frac{1}{1-t}, \frac{1}{1+t}, e^{-t} \rangle$ is $D_r = \{t \in \mathbb{R} \mid t \neq \pm 1\}$.

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  1. We project into the xy-plane by setting $z(t) = t = 0$. Observe that as t increases, the curve $\mathbf{r}(t)$ moves clockwise (since $\mathbf{r}\left(\frac{\pi}{2}\right) = \langle -1, 0, 0 \rangle$) from the point $(0, 1, 0)$ and traces out the unit circle in the xy-plane.

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  2. We project into the xz-plane by setting $y(t) = \cos t = 0$. Observe that as t increases, $\mathbf{r}(t)$ traces the curve $-\sin t$ vertically in the xz-plane.
  3. We project into the yz-plane by setting $x(t) = -\sin t = 0$. Observe that as t increases, $\mathbf{r}(t)$ traces the curve $\cos t$ vertically in the yz-plane.

Parametrization of Curves in $\mathbb{R}^3$

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$$z(t) = z = x^2 - y^2 + 1 = 4\cos^2 t - 4\sin^2 t + 1 = 4\cos(2t) + 1.$$

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$$z(t) = z = x^2 - y^2 + 1 = 4\cos^2 t - 4\sin^2 t + 1 = 4\cos(2t) + 1.$$

Ultimately, we have that $\mathcal{C}$ is given by the vector-valued function

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\cos(2t) + 1 \rangle.$$

Embedding a Circle in $\mathbb{R}^3$, Pt. I

Choose the vector-valued function that parametrizes the circle of radius 3 centered at $(2, 6, 8)$ and parallel to the *xy*-plane.

(a.) $\mathbf{r}(t) = \langle \sqrt{t-2}, \sqrt{t-6}, t-8 \rangle$

(b.) $\mathbf{s}(t) = \langle 3\cos t + 2, 3\sin t + 6, 8 \rangle$

(c.) $\mathbf{u}(t) = \langle 2\cos t, \sin t + 6, 8 \rangle$

(d.) $\mathbf{v}(t) = 3\mathbf{r}(t)$

Embedding a Circle in $\mathbb{R}^3$, Pt. II

Choose the vector-valued function that parametrizes the circle of radius 3 centered at $(2, 6, 8)$ and parallel to the *xz*-plane.

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(b.) $\mathbf{s}(t) = \langle 3\cos t + 2, 6, 3\sin t + 8 \rangle$

(c.) $\mathbf{u}(t) = \langle 2\cos t, 6, \sin t + 8 \rangle$

(d.) $\mathbf{v}(t) = 3\mathbf{r}(t)$

Embedding an Ellipse in $\mathbb{R}^3$

Choose the vector-valued function that parametrizes the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ centered at $(9, -4, 0)$ and parallel to the xy-plane.

(a.) $\mathbf{r}(t) = \langle \frac{\sqrt{t-9}}{2}, \frac{\sqrt{t+4}}{3}, 0 \rangle$

(c.) $\mathbf{u}(t) = \langle 2\cos t + 9, 3\sin t - 4, 0 \rangle$

(b.) $\mathbf{s}(t) = \langle \frac{\sqrt{t-9}}{2}, \frac{\sqrt{t+4}}{3}, 1 \rangle$

(d.) $\mathbf{v}(t) = \langle 2\cos(t-9), 3\sin(t+4), 1 \rangle$

Calculus of Vector-Valued Functions

- Considering that vector-valued functions consist of components that are functions of a single variable, we can extend many of the notions of Calculus I to vector-valued functions.

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$$\lim_{t \to t_0} \mathbf{r}(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle.$$

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- Given a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, we have that

$$\frac{d}{dt}\mathbf{r}(t) = \left\langle \frac{d}{dt}x(t), \frac{d}{dt}y(t), \frac{d}{dt}z(t) \right\rangle.$$