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- We refer to the point (r, θ) as the **polar coordinates** analog of the point $(x = r\cos\theta, y = r\sin\theta)$ in Cartesian coordinates.

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- Generally, we can obtain a point in polar coordinates from any pair (x, y) in Cartesian (or rectangular) coordinates by taking

$$r = \sqrt{x^2 + y^2} \qquad \text{and}$$

$$\theta = \text{ptan}^{-1}\left(\frac{y}{x}\right) = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x > 0; \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & \text{if } x < 0; \text{ and} \\ \pm\frac{\pi}{2} & \text{if } x = 0. \end{cases}$$

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- **Caution:** By definition, the range of the function $\tan^{-1}(-)$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$, hence we must be careful when computing θ from (x, y).

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Recognizing Graphs of Polar Equations

Describe the graph of the polar equation $r = 2a\cos\theta$.

(a.) cardioid symmetric across x-axis

(b.) cardioid symmetric across y-axis

(c.) circle centered at $(a, 0)$

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We have that $x^2 + y^2 = r^2 = 2a(r\cos\theta) = 2ax$ after multiplying both sides of the given equation by r. By subtracting $2ax$ from both sides and completing the square, we have that $(x - a)^2 + y^2 = a^2$.

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Plot a few points, and observe the shape of the graph.

Cylindrical Coordinates

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$$(x, y, z) \leftrightarrow (r, \theta, z)$$

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\theta \qquad \theta = \text{ptan}^{-1}\left(\tfrac{y}{x}\right)$$

$$z = z \qquad z = z$$

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True (a.) or False (b.)

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The cylindrical equation $r\cos\theta = 0$ defines the yz-plane.

(a.) True. We have that $x = r\cos\theta = 0$. Every point of the form $(0, y, z)$ satisfies this equation, and the yz-plane consists of precisely those points.

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$$(x, y, z) \leftrightarrow (\rho, \theta, \phi)$$

$$x = \rho \sin\phi \cos\theta \qquad \rho = \sqrt{x^2 + y^2 + z^2}$$

$$y = \rho \sin\phi \sin\theta \qquad \theta = \mathrm{ptan}^{-1}\left(\frac{y}{x}\right)$$

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  1. Given that $\phi_0 = 0$, the equation $\phi = 0$ gives the positive z-axis.
  2. Given that $\phi_0 = \frac{\pi}{2}$, the equation $\phi = \frac{\pi}{2}$ gives the xy-plane.
  3. Given that $\phi_0 = \pi$, the equation $\phi = \pi$ gives the negative z-axis.

True (a.) or False (b.)

The set of points satisfying $0 \leq \rho \leq 3$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \frac{\pi}{2}$ defines the top half of a solid ball of radius 3 centered at the origin.

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(a.) True. Rho (ρ) gives the radius, which is at most 3; theta (θ) gives the angle of revolution, which is at most 2π (one revolution); and phi (ϕ) gives the angle of declination from the *z*-axis, which is at most $\frac{\pi}{2}$ (or 90°).