Basic Properties of Vectors

- Each vector in $\mathbb{R}^n$ is uniquely determined by two points $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$ by identifying the ray $\overrightarrow{PQ}$ with the vector $\mathbf{v} = \langle q_1 - p_1, \ldots, q_n - p_n \rangle$ based at the origin.

Basic Properties of Vectors

- Each vector in $\mathbb{R}^n$ is uniquely determined by two points $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$ by identifying the ray $\overrightarrow{PQ}$ with the vector $\mathbf{v} = \langle q_1 - p_1, \ldots, q_n - p_n \rangle$ based at the origin.

- Each vector $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ in $\mathbb{R}^n$ possesses a **magnitude** (or length)

$$||\mathbf{v}|| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Basic Properties of Vectors

- Each vector in $\mathbb{R}^n$ is uniquely determined by two points $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$ by identifying the ray $\overrightarrow{PQ}$ with the vector $\mathbf{v} = \langle q_1 - p_1, \ldots, q_n - p_n \rangle$ based at the origin.

- Each vector $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ in $\mathbb{R}^n$ possesses a **magnitude** (or length)

$$||\mathbf{v}|| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

- Given vectors $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ and $\mathbf{w} = \langle w_1, \ldots, w_n \rangle$ in $\mathbb{R}^n$, we define their sum to be the vector $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, \ldots, v_n + w_n \rangle$.

Basic Properties of Vectors

- Each vector in $\mathbb{R}^n$ is uniquely determined by two points $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$ by identifying the ray $\overrightarrow{PQ}$ with the vector $\mathbf{v} = \langle q_1 - p_1, \ldots, q_n - p_n \rangle$ based at the origin.
- Each vector $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ in $\mathbb{R}^n$ possesses a **magnitude** (or length)

$$||\mathbf{v}|| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

- Given vectors $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ and $\mathbf{w} = \langle w_1, \ldots, w_n \rangle$ in $\mathbb{R}^n$, we define their sum to be the vector $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, \ldots, v_n + w_n \rangle$.
- Given a vector $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ in $\mathbb{R}^n$ and a **scalar** λ in $\mathbb{R}$, we define the scalar product of $\mathbf{v}$ by λ to be the vector $\lambda\mathbf{v} = \langle \lambda v_1, \ldots, \lambda v_n \rangle$.

Computation with Vectors

Consider the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 0, 1 \rangle$ in $\mathbb{R}^3$. Compute the vectors $3\mathbf{v}$, $-\mathbf{w}$, and $3\mathbf{v} - \mathbf{w}$; then, determine the magnitude $||3\mathbf{v} - \mathbf{w}||$ of the vector $3\mathbf{v} - \mathbf{w}$.

Computation with Vectors

Consider the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 0, 1 \rangle$ in $\mathbb{R}^3$. Compute the vectors $3\mathbf{v}$, $-\mathbf{w}$, and $3\mathbf{v} - \mathbf{w}$; then, determine the magnitude $||3\mathbf{v} - \mathbf{w}||$ of the vector $3\mathbf{v} - \mathbf{w}$.

(a.) By definition of scalar multiplication, we have that $3\mathbf{v} = 3\langle 1, 2, 3 \rangle = \langle 3, 6, 9 \rangle$ and $-\mathbf{w} = -\langle -1, 0, 1 \rangle = \langle 1, 0, -1 \rangle$.

Computation with Vectors

Consider the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 0, 1 \rangle$ in $\mathbb{R}^3$. Compute the vectors $3\mathbf{v}$, $-\mathbf{w}$, and $3\mathbf{v} - \mathbf{w}$; then, determine the magnitude $||3\mathbf{v} - \mathbf{w}||$ of the vector $3\mathbf{v} - \mathbf{w}$.

(a.) By definition of scalar multiplication, we have that
$3\mathbf{v} = 3\langle 1, 2, 3 \rangle = \langle 3, 6, 9 \rangle$ and $-\mathbf{w} = -\langle -1, 0, 1 \rangle = \langle 1, 0, -1 \rangle$.
Consequently, it follows that $3\mathbf{v} - \mathbf{w} = \langle 3, 6, 9 \rangle + \langle 1, 0, -1 \rangle = \langle 4, 6, 8 \rangle$.

Computation with Vectors

Consider the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 0, 1 \rangle$ in $\mathbb{R}^3$. Compute the vectors $3\mathbf{v}$, $-\mathbf{w}$, and $3\mathbf{v} - \mathbf{w}$; then, determine the magnitude $||3\mathbf{v} - \mathbf{w}||$ of the vector $3\mathbf{v} - \mathbf{w}$.

(a.) By definition of scalar multiplication, we have that $3\mathbf{v} = 3\langle 1, 2, 3 \rangle = \langle 3, 6, 9 \rangle$ and $-\mathbf{w} = -\langle -1, 0, 1 \rangle = \langle 1, 0, -1 \rangle$. Consequently, it follows that $3\mathbf{v} - \mathbf{w} = \langle 3, 6, 9 \rangle + \langle 1, 0, -1 \rangle = \langle 4, 6, 8 \rangle$. Ultimately, we find that $||3\mathbf{v} - \mathbf{w}|| = \sqrt{4^2 + 6^2 + 8^2} = \sqrt{116} = 2\sqrt{29}$.

Basic Properties of Vectors

- **Unit vectors** are those vectors $\mathbf{v}$ such that $||\mathbf{v}|| = 1$.

Basic Properties of Vectors

- **Unit vectors** are those vectors $\mathbf{v}$ such that $||\mathbf{v}|| = 1$. Given a nonzero vector $\mathbf{w}$, we may scale $\mathbf{w}$ to obtain a unit vector $\frac{1}{||\mathbf{w}||}\mathbf{w}$.

Basic Properties of Vectors

- **Unit vectors** are those vectors $\mathbf{v}$ such that $||\mathbf{v}|| = 1$. Given a nonzero vector $\mathbf{w}$, we may scale $\mathbf{w}$ to obtain a unit vector $\frac{1}{||\mathbf{w}||}\mathbf{w}$.
- We refer to the vector $\mathbf{e}_i$ with 1 in the ith place and 0 elsewhere as the ith **standard basis vector** of $\mathbb{R}^n$.

Basic Properties of Vectors

- **Unit vectors** are those vectors $\mathbf{v}$ such that $||\mathbf{v}|| = 1$. Given a nonzero vector $\mathbf{w}$, we may scale $\mathbf{w}$ to obtain a unit vector $\frac{1}{||\mathbf{w}||}\mathbf{w}$.
- We refer to the vector $\mathbf{e}_i$ with 1 in the ith place and 0 elsewhere as the ith **standard basis vector** of $\mathbb{R}^n$. For example, the standard basis vectors of $\mathbb{R}^3$ are $\mathbf{e}_1 = \langle 1, 0, 0 \rangle$, $\mathbf{e}_2 = \langle 0, 1, 0 \rangle$, and $\mathbf{e}_3 = \langle 0, 0, 1 \rangle$.

Basic Properties of Vectors

- **Unit vectors** are those vectors $\mathbf{v}$ such that $||\mathbf{v}|| = 1$. Given a nonzero vector $\mathbf{w}$, we may scale $\mathbf{w}$ to obtain a unit vector $\frac{1}{||\mathbf{w}||}\mathbf{w}$.
- We refer to the vector $\mathbf{e}_i$ with 1 in the ith place and 0 elsewhere as the ith **standard basis vector** of $\mathbb{R}^n$. For example, the standard basis vectors of $\mathbb{R}^3$ are $\mathbf{e}_1 = \langle 1, 0, 0 \rangle$, $\mathbf{e}_2 = \langle 0, 1, 0 \rangle$, and $\mathbf{e}_3 = \langle 0, 0, 1 \rangle$.
- Given two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$, we have the **Triangle Inequality**

$$||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$$

  with equality if and only if one of $\mathbf{v}$ or $\mathbf{w}$ is 0 <u>or</u> $\mathbf{v} = \lambda \mathbf{w}$ $(\lambda > 0)$.

Basic Properties of Vectors

- Each line in $\mathbb{R}^n$ is uniquely determined by a point $P = (p_1, \ldots, p_n)$ and a directional vector $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$.

Basic Properties of Vectors

- Each line in $\mathbb{R}^n$ is uniquely determined by a point $P = (p_1, \ldots, p_n)$ and a directional vector $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$. Explicitly, the line ℓ through P in the direction of $\mathbf{v}$ is given by the parametric equation

$$\mathbf{r}(t) = \langle P \rangle + t\mathbf{v} = \langle p_1 + tv_1, \ldots, p_n + tv_n \rangle,$$

where $\langle P \rangle = \langle p_1, \ldots, p_n \rangle$ and t is a real number in $(-\infty, \infty)$.

Basic Properties of Vectors

- Each line in $\mathbb{R}^n$ is uniquely determined by a point $P = (p_1, \ldots, p_n)$ and a directional vector $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$. Explicitly, the line ℓ through P in the direction of $\mathbf{v}$ is given by the parametric equation

$$\mathbf{r}(t) = \langle P \rangle + t\mathbf{v} = \langle p_1 + tv_1, \ldots, p_n + tv_n \rangle,$$

  where $\langle P \rangle = \langle p_1, \ldots, p_n \rangle$ and t is a real number in $(-\infty, \infty)$.

- Given vectors $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ and $\mathbf{w} = \langle w_1, \ldots, w_n \rangle$ in $\mathbb{R}^n$, we define the **dot product** of $\mathbf{v}$ and $\mathbf{w}$ as the scalar $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n$.

Basic Properties of Vectors

- Each line in $\mathbb{R}^n$ is uniquely determined by a point $P = (p_1, \ldots, p_n)$ and a directional vector $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$. Explicitly, the line ℓ through P in the direction of $\mathbf{v}$ is given by the parametric equation

$$\mathbf{r}(t) = \langle P \rangle + t\mathbf{v} = \langle p_1 + tv_1, \ldots, p_n + tv_n \rangle,$$

  where $\langle P \rangle = \langle p_1, \ldots, p_n \rangle$ and t is a real number in $(-\infty, \infty)$.

- Given vectors $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ and $\mathbf{w} = \langle w_1, \ldots, w_n \rangle$ in $\mathbb{R}^n$, we define the **dot product** of $\mathbf{v}$ and $\mathbf{w}$ as the <u>scalar</u> $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n$.

- Given vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ with angle θ between them, we have that

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| \, ||\mathbf{w}|| \cos\theta.$$

Basic Properties of Vectors

- Each line in $\mathbb{R}^n$ is uniquely determined by a point $P = (p_1, \ldots, p_n)$ and a directional vector $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$. Explicitly, the line ℓ through P in the direction of $\mathbf{v}$ is given by the parametric equation

$$\mathbf{r}(t) = \langle P \rangle + t\mathbf{v} = \langle p_1 + tv_1, \ldots, p_n + tv_n \rangle,$$

  where $\langle P \rangle = \langle p_1, \ldots, p_n \rangle$ and t is a real number in $(-\infty, \infty)$.

- Given vectors $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ and $\mathbf{w} = \langle w_1, \ldots, w_n \rangle$ in $\mathbb{R}^n$, we define the **dot product** of $\mathbf{v}$ and $\mathbf{w}$ as the scalar $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n$.

- Given vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ with angle θ between them, we have that

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| \, ||\mathbf{w}|| \cos\theta.$$

  We say that $\mathbf{v}$ and $\mathbf{w}$ are **orthogonal** if $\mathbf{v} \cdot \mathbf{w} = 0$.

Basic Properties of Vectors

- Each line in $\mathbb{R}^n$ is uniquely determined by a point $P = (p_1, \ldots, p_n)$ and a directional vector $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$. Explicitly, the line ℓ through P in the direction of $\mathbf{v}$ is given by the parametric equation

$$\mathbf{r}(t) = \langle P \rangle + t\mathbf{v} = \langle p_1 + tv_1, \ldots, p_n + tv_n \rangle,$$

  where $\langle P \rangle = \langle p_1, \ldots, p_n \rangle$ and t is a real number in $(-\infty, \infty)$.

- Given vectors $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ and $\mathbf{w} = \langle w_1, \ldots, w_n \rangle$ in $\mathbb{R}^n$, we define the **dot product** of $\mathbf{v}$ and $\mathbf{w}$ as the <u>scalar</u> $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n$.

- Given vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ with angle θ between them, we have that

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| \, ||\mathbf{w}|| \cos \theta.$$

  We say that $\mathbf{v}$ and $\mathbf{w}$ are **orthogonal** if $\mathbf{v} \cdot \mathbf{w} = 0$. Observe that in $\mathbb{R}^2$ or $\mathbb{R}^3$, this is equivalent to the notion of "perpendicular" vectors.

True (a.) or False (b.)

Given that two lines ℓ_1 and ℓ_2 are both parallel to the line ℓ_3, it must be true that the lines ℓ_1 and ℓ_2 are parallel.

True (a.) or False (b.)

Given that two lines ℓ_1 and ℓ_2 are both parallel to the line ℓ_3, it must be true that the lines ℓ_1 and ℓ_2 are parallel.

(a.) True. Lines are uniquely determined by a point P and a directional vector $\mathbf{v}$. Parallel lines have the same (up to a scalar multiple) directional vector, so the directional vector of ℓ_1 and ℓ_2 must be the same.

True (a.) or False (b.)

Given that two lines ℓ_1 and ℓ_2 are both orthogonal to the line ℓ_3, it must be true that the lines ℓ_1 and ℓ_2 are orthogonal.

True (a.) or False (b.)

Given that two lines ℓ_1 and ℓ_2 are both orthogonal to the line ℓ_3, it must be true that the lines ℓ_1 and ℓ_2 are orthogonal.

b.) False. For example, ℓ_1 and ℓ_2 could be skew. Consider the lines $\ell_1 = t\langle 1, 1, 1\rangle$, $\ell_2 = t\langle 2, 1, 2\rangle$, and $\ell_3 = t\langle 1, 0, -1\rangle$. We have that

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_3 &= \langle 1, 1, 1\rangle \cdot \langle 1, 0, -1\rangle = 0 \text{ and} \\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= \langle 2, 1, 2\rangle \cdot \langle 1, 0, -1\rangle = 0 \text{ but} \\ \mathbf{v}_1 \cdot \mathbf{v}_2 &= \langle 1, 1, 1\rangle \cdot \langle 2, 1, 2\rangle = 5, \end{aligned}$$

hence ℓ_1 and ℓ_2 are not orthogonal, as their directional vectors are not.

Basic Properties of Vectors

True (a.) or False (b.)

Given that $\mathbf{v} \cdot \mathbf{w} < 0$, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is acute.

True (a.) or False (b.)

Given that $\mathbf{v} \cdot \mathbf{w} < 0$, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is acute.

b.) False. Combining the formula $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\,||\mathbf{w}|| \cos\theta$ with the fact that $||\mathbf{v}||$ and $||\mathbf{w}||$ are by definition positive, we conclude that $\cos\theta < 0$. Considering that $\cos\theta \geq 0$ whenever $0 \leq \theta \leq \frac{\pi}{2}$ or $\frac{3\pi}{2} \leq \theta < 2\pi$, we must have that $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, from which it follow that θ is obtuse.

Basic Properties of Vectors

True (a.) or False (b.)

Given that $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{u}$ is orthogonal to $\mathbf{v} + \mathbf{w}$.

True (a.) or False (b.)

Given that $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{u}$ is orthogonal to $\mathbf{v} + \mathbf{w}$.

a.) True. Explicitly, we will assume that $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$, $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$, and $\mathbf{w} = \langle w_1, \ldots, w_n \rangle$. We are given that $u_1 v_1 + \cdots + u_n v_n = \mathbf{u} \cdot \mathbf{v} = 0$ and $u_1 w_1 + \cdots + u_n w_n = \mathbf{u} \cdot \mathbf{w} = 0$. Considering that $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, \ldots, v_n + w_n \rangle$, we have that

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= u_1 \langle v_1 + w_1 \rangle + \cdots + u_n \langle v_n + w_n \rangle \\ &= \langle u_1 v_1 + \cdots u_n v_n \rangle + \langle u_1 w_1 + \cdots + u_n w_n \rangle = 0. \end{aligned}$$

Basic Properties of Vectors

- We define a real $n \times n$ **matrix** to be an $n \times n$ array of real numbers. Recall that the sum of two $n \times n$ matrices $[a_{ij}]$ and $[b_{ij}]$ is defined to be the matrix $[s_{ij}]$ such that $s_{ij} = a_{ij} + b_{ij}$, and the product of the same $n \times n$ matrices is defined to be the matrix $[p_{ij}]$ such that

$$p_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Basic Properties of Vectors

- We define a real $n \times n$ **matrix** to be an $n \times n$ array of real numbers. Recall that the sum of two $n \times n$ matrices $[a_{ij}]$ and $[b_{ij}]$ is defined to be the matrix $[s_{ij}]$ such that $s_{ij} = a_{ij} + b_{ij}$, and the product of the same $n \times n$ matrices is defined to be the matrix $[p_{ij}]$ such that

$$p_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

  Put another way, the entry of $[p_{ij}]$ in the ith row and jth column is the sum of the products of the entries a_{ik} in the ith row and kth column of $[a_{ij}]$ and b_{kj} in the kth row and jth column of $[b_{ij}]$.

Basic Properties of Vectors

- We define a real $n \times n$ **matrix** to be an $n \times n$ array of real numbers. Recall that the sum of two $n \times n$ matrices $[a_{ij}]$ and $[b_{ij}]$ is defined to be the matrix $[s_{ij}]$ such that $s_{ij} = a_{ij} + b_{ij}$, and the product of the same $n \times n$ matrices is defined to be the matrix $[p_{ij}]$ such that

$$p_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

  Put another way, the entry of $[p_{ij}]$ in the ith row and jth column is the sum of the products of the entries a_{ik} in the ith row and kth column of $[a_{ij}]$ and b_{kj} in the kth row and jth column of $[b_{ij}]$.

- Given a 2×2 matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, the **determinant** of A is the scalar

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Computation with Matrices

Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Compute the matrices $A + B$ and AB; then, find the determinant of $A + B$.

Computation with Matrices

Consider the matrices $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Compute the matrices $A + B$ and AB; then, find the determinant of $A + B$.

By definition of matrix addition and multiplication, we have that

$$A + B = \begin{bmatrix} 1+1 & 2+0 \\ 3+0 & 4-1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 0 & 1 \cdot 0 + 2 \cdot -1 \\ 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 0 + 4 \cdot -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}.$$

Computation with Matrices

Consider the matrices $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Compute the matrices $A + B$ and AB; then, find the determinant of $A + B$.

By definition of matrix addition and multiplication, we have that

$$A + B = \begin{bmatrix} 1+1 & 2+0 \\ 3+0 & 4-1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 0 & 1 \cdot 0 + 2 \cdot -1 \\ 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 0 + 4 \cdot -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}.$$

Consequently, we have that $\det(A + B) = 2 \cdot 3 - 2 \cdot 3 = 0$.

Basic Properties of Vectors

- Given vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ in $\mathbb{R}^3$, we define the **cross product** of $\mathbf{v}$ and $\mathbf{w}$ to be the vector

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{e}_3.$$

- Given vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ in $\mathbb{R}^3$, we define the **cross product** of $\mathbf{v}$ and $\mathbf{w}$ to be the vector

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{e}_3.$$

One method of computing the cross product is to write the array

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_1 & \mathbf{e}_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \\ w_1 & w_2 & w_3 & w_1 & w_2 \end{vmatrix};$$

add the three top-left-to-bottom-right full diagonals; and subtract the top-right-to-bottom-left full diagonals.

Basic Properties of Vectors

- Given vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ in $\mathbb{R}^3$, we define the **cross product** of $\mathbf{v}$ and $\mathbf{w}$ to be the vector

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{e}_3.$$

One method of computing the cross product is to write the array

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_1 & \mathbf{e}_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \\ w_1 & w_2 & w_3 & w_1 & w_2 \end{vmatrix};$$

add the three top-left-to-bottom-right full diagonals; and subtract the top-right-to-bottom-left full diagonals. From this, one will obtain

$$\begin{aligned} \mathbf{v} \times \mathbf{w} = \mathbf{e}_1 v_2 w_3 + \mathbf{e}_2 v_3 w_1 + \mathbf{e}_3 v_1 w_2 \\ - \mathbf{e}_1 v_3 w_2 - \mathbf{e}_2 v_1 w_3 - \mathbf{e}_3 v_2 w_1. \end{aligned}$$

Basic Properties of Vectors

- Recall the following properties of the cross product.
  1. $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

Basic Properties of Vectors

- Recall the following properties of the cross product.
  1. $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
  2. $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}|| \, ||\mathbf{w}|| \sin\theta$ for the angle θ between $\mathbf{v}$ and $\mathbf{w}$.

Basic Properties of Vectors

- Recall the following properties of the cross product.
  1. $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
  2. $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||\,||\mathbf{w}|| \sin\theta$ for the angle θ between $\mathbf{v}$ and $\mathbf{w}$.
  3. $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ whenever $\mathbf{w} = \lambda\mathbf{v}$ for some scalar λ.

Basic Properties of Vectors

- Recall the following properties of the cross product.
  1. $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
  2. $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||\,||\mathbf{w}|| \sin\theta$ for the angle θ between $\mathbf{v}$ and $\mathbf{w}$.
  3. $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ whenever $\mathbf{w} = \lambda\mathbf{v}$ for some scalar λ.
  4. $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$, i.e., the cross product is anticommutative.

Basic Properties of Vectors

- Recall the following properties of the cross product.
  1. $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
  2. $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}|| \, ||\mathbf{w}|| \sin\theta$ for the angle θ between $\mathbf{v}$ and $\mathbf{w}$.
  3. $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ whenever $\mathbf{w} = \lambda\mathbf{v}$ for some scalar λ.
  4. $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$, i.e., the cross product is anticommutative.
  5. $(\lambda\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (\lambda\mathbf{w}) = \lambda(\mathbf{v} \times \mathbf{w})$ for any scalar λ.

Basic Properties of Vectors

- Recall the following properties of the cross product.
  1. $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
  2. $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||\,||\mathbf{w}||\sin\theta$ for the angle θ between $\mathbf{v}$ and $\mathbf{w}$.
  3. $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ whenever $\mathbf{w} = \lambda\mathbf{v}$ for some scalar λ.
  4. $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$, i.e., the cross product is anticommutative.
  5. $(\lambda\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (\lambda\mathbf{w}) = \lambda(\mathbf{v} \times \mathbf{w})$ for any scalar λ.
  6. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$ and $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

Basic Properties of Vectors

- Recall the following properties of the cross product.
  1. $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
  2. $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||\,||\mathbf{w}|| \sin\theta$ for the angle θ between $\mathbf{v}$ and $\mathbf{w}$.
  3. $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ whenever $\mathbf{w} = \lambda\mathbf{v}$ for some scalar λ.
  4. $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$, i.e., the cross product is anticommutative.
  5. $(\lambda\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (\lambda\mathbf{w}) = \lambda(\mathbf{v} \times \mathbf{w})$ for any scalar λ.
  6. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$ and $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
  7. $||\mathbf{v} \times \mathbf{w}||^2 = ||\mathbf{v}||^2||\mathbf{w}||^2 - (\mathbf{v} \cdot \mathbf{w})^2$

Basic Properties of Vectors

- Recall the following properties of the cross product.
  1. $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.
  2. $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||\,||\mathbf{w}|| \sin\theta$ for the angle θ between $\mathbf{v}$ and $\mathbf{w}$.
  3. $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ whenever $\mathbf{w} = \lambda\mathbf{v}$ for some scalar λ.
  4. $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$, i.e., the cross product is anticommutative.
  5. $(\lambda\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (\lambda\mathbf{w}) = \lambda(\mathbf{v} \times \mathbf{w})$ for any scalar λ.
  6. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$ and $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
  7. $||\mathbf{v} \times \mathbf{w}||^2 = ||\mathbf{v}||^2||\mathbf{w}||^2 - (\mathbf{v} \cdot \mathbf{w})^2$

Properties (5.) and (6.) imply that the cross product is **bilinear**.

Planes in $\mathbb{R}^3$

- Each line in $\mathbb{R}^2$ is determined uniquely (up to a scalar multiple) by a linear equation $ax + by = c$.

Planes in $\mathbb{R}^3$

- Each line in $\mathbb{R}^2$ is determined uniquely (up to a scalar multiple) by a linear equation $ax + by = c$. Likewise, each plane in $\mathbb{R}^3$ is determined uniquely (up to a scalar multiple) by an equation $ax + by + cz = d$.

Planes in $\mathbb{R}^3$

- Each line in $\mathbb{R}^2$ is determined uniquely (up to a scalar multiple) by a linear equation $ax + by = c$. Likewise, each plane in $\mathbb{R}^3$ is determined uniquely (up to a scalar multiple) by an equation $ax + by + cz = d$.

- Explicitly, the plane $\mathcal{P}$ through the point $P_0 = (x_0, y_0, z_0)$ is uniquely determined (up to a scalar multiple) by a **normal** vector $\mathbf{n} = \langle a, b, c\rangle$ according to the following: a point P lies on $\mathcal{P}$ if and only if $\mathbf{n}$ and $\overrightarrow{P_0P}$ are orthogonal if and only if $n \cdot \overrightarrow{P_0P} = 0$ if and only if

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Planes in $\mathbb{R}^3$

- Each line in $\mathbb{R}^2$ is determined uniquely (up to a scalar multiple) by a linear equation $ax + by = c$. Likewise, each plane in $\mathbb{R}^3$ is determined uniquely (up to a scalar multiple) by an equation $ax + by + cz = d$.

- Explicitly, the plane $\mathcal{P}$ through the point $P_0 = (x_0, y_0, z_0)$ is uniquely determined (up to a scalar multiple) by a **normal** vector $\mathbf{n} = \langle a, b, c\rangle$ according to the following: a point P lies on $\mathcal{P}$ if and only if $\mathbf{n}$ and $\overrightarrow{P_0P}$ are orthogonal if and only if $n \cdot \overrightarrow{P_0P} = 0$ if and only if

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

By setting $d = ax_0 + by_0 + cz_0$, we have $ax + by + cz = d$.

Planes in $\mathbb{R}^3$

- Each line in $\mathbb{R}^2$ is determined uniquely (up to a scalar multiple) by a linear equation $ax + by = c$. Likewise, each plane in $\mathbb{R}^3$ is determined uniquely (up to a scalar multiple) by an equation $ax + by + cz = d$.

- Explicitly, the plane $\mathcal{P}$ through the point $P_0 = (x_0, y_0, z_0)$ is uniquely determined (up to a scalar multiple) by a **normal** vector $\mathbf{n} = \langle a, b, c \rangle$ according to the following: a point P lies on $\mathcal{P}$ if and only if $\mathbf{n}$ and $\overrightarrow{P_0P}$ are orthogonal if and only if $n \cdot \overrightarrow{P_0P} = 0$ if and only if

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

  By setting $d = ax_0 + by_0 + cz_0$, we have $ax + by + cz = d$.

- Given three points $P = (x_0, y_0, z_0)$, Q, and R, the equation of the plane through P, Q, and R can be determined by setting $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ and computing the dot product $d = \mathbf{n} \cdot \langle x_0, y_0, z_0 \rangle$.

Planes in $\mathbb{R}^3$

True (a.) or False (b.)

Given that two lines ℓ_1 and ℓ_2 are both parallel to the plane $\mathcal{P}$, it must be true that the lines ℓ_1 and ℓ_2 are parallel.

True (a.) or False (b.)

Given that two lines ℓ_1 and ℓ_2 are both parallel to the plane $\mathcal{P}$, it must be true that the lines ℓ_1 and ℓ_2 are parallel.

b.) False. Lines that are parallel to the plane $\mathcal{P}$ are orthogonal to the normal vector $\mathbf{n}$ that defines $\mathcal{P}$. Consequently, the statement in question is a reformulation of the previous false statement.

Planes in $\mathbb{R}^3$

True (a.) or False (b.)

Given that two lines ℓ_1 and ℓ_2 are both orthogonal to the plane $\mathcal{P}$, it must be true that the lines ℓ_1 and ℓ_2 are parallel.

True (a.) or False (b.)

Given that two lines ℓ_1 and ℓ_2 are both orthogonal to the plane $\mathcal{P}$, it must be true that the lines ℓ_1 and ℓ_2 are parallel.

a.) True. Lines that are orthogonal to the plane $\mathcal{P}$ are parallel to the normal vector $\mathbf{n}$ that defines $\mathcal{P}$. Consequently, the statement in question is a reformulation of the first true statement.

Planes in $\mathbb{R}^3$

True (a.) or False (b.)

The plane $\mathcal{P}$ defined by the equation $x + y = 0$ contains the z-axis.

Planes in $\mathbb{R}^3$

True (a.) or False (b.)

The plane $\mathcal{P}$ defined by the equation $x + y = 0$ contains the z-axis.

a.) True. We note that the normal vector determining the given plane is $\mathbf{n} = \langle 1, 1, 0 \rangle$. Consequently, the vector $\mathbf{e}_3 = \langle 0, 0, 1 \rangle$ that determines the z-axis is orthogonal to $\mathbf{n}$ and must therefore be contained in $\mathcal{P}$.

True (a.) or False (b.)

The plane $\mathcal{P}$ defined by the equation $x + y = 0$ contains the z-axis.

a.) True. We note that the normal vector determining the given plane is $\mathbf{n} = \langle 1, 1, 0 \rangle$. Consequently, the vector $\mathbf{e}_3 = \langle 0, 0, 1 \rangle$ that determines the z-axis is orthogonal to $\mathbf{n}$ and must therefore be contained in $\mathcal{P}$.

One other way to see it is that all points of the form $(0, 0, z)$ satisfy the given equation and must therefore lie in the given plane.

True (a.) or False (b.)

The plane $\mathcal{P}_1$ defined by the equation $x + y + z = 1$ and the plane $\mathcal{P}_2$ defined by the equation $x + 2y + 3z = 1$ intersect.

True (a.) or False (b.)

The plane $\mathcal{P}_1$ defined by the equation $x + y + z = 1$ and the plane $\mathcal{P}_2$ defined by the equation $x + 2y + 3z = 1$ intersect.

a.) True. Both planes contain the point $(1, 0, 0)$ and therefore intersect at this point. Further, we find that the vector $\mathbf{v} = \langle 1, -2, 1 \rangle$ is orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Consequently, the line of intersection is given by

$$\mathbf{r}(t) = \langle 1, 0, 0 \rangle + t\langle 1, -2, 1 \rangle = \langle t + 1, -2t, t \rangle.$$

Planes in $\mathbb{R}^3$

True (a.) or False (b.)

The plane $\mathcal{P}_1$ defined by the equation $x + y + z = 1$ and the plane $\mathcal{P}_2$ defined by the equation $x + 2y + 3z = 1$ intersect.

a.) True. Both planes contain the point $(1, 0, 0)$ and therefore intersect at this point. Further, we find that the vector $\mathbf{v} = \langle 1, -2, 1 \rangle$ is orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Consequently, the line of intersection is given by

$$\mathbf{r}(t) = \langle 1, 0, 0 \rangle + t\langle 1, -2, 1 \rangle = \langle t + 1, -2t, t \rangle.$$

One other way to see it is to solve the given system of equations. We find that $y + z = 2y + 3z$ so that $y = -2z$. Plugging this back into the original equation gives $x - z = 1$ so that $x = z + 1$. Bada-bing, bada-boom.

Vector Spaces and Linear Transformations

- We refer to $\mathbb{R}^n$ as a **vector space** over $\mathbb{R}$. Generally, a vector space over $\mathbb{R}$ is an algebraic structure whose objects are vectors and whose operations are vector addition and scalar multiplication.

Vector Spaces and Linear Transformations

- We refer to $\mathbb{R}^n$ as a **vector space** over $\mathbb{R}$. Generally, a vector space over $\mathbb{R}$ is an algebraic structure whose objects are vectors and whose operations are vector addition and scalar multiplication.

- We say that the **dimension** of a vector space is the (unique) number of vectors in a basis. For instance, the dimension of $\mathbb{R}^n$ is n since the n vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a basis for $\mathbb{R}^n$.

Vector Spaces and Linear Transformations

- We refer to $\mathbb{R}^n$ as a **vector space** over $\mathbb{R}$. Generally, a vector space over $\mathbb{R}$ is an algebraic structure whose objects are vectors and whose operations are vector addition and scalar multiplication.
- We say that the **dimension** of a vector space is the (unique) number of vectors in a basis. For instance, the dimension of $\mathbb{R}^n$ is n since the n vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a basis for $\mathbb{R}^n$.
- Functions between vector spaces are called **linear transformations**. Given vector spaces V and W, a linear transformation $T : V \to W$ must satisfy $T(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2) = \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2)$. Cross product with a fixed vector is an example of a linear transformation.

Vector Spaces and Linear Transformations

- We refer to $\mathbb{R}^n$ as a **vector space** over $\mathbb{R}$. Generally, a vector space over $\mathbb{R}$ is an algebraic structure whose objects are vectors and whose operations are vector addition and scalar multiplication.
- We say that the **dimension** of a vector space is the (unique) number of vectors in a basis. For instance, the dimension of $\mathbb{R}^n$ is n since the n vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a basis for $\mathbb{R}^n$.
- Functions between vector spaces are called **linear transformations**. Given vector spaces V and W, a linear transformation $T : V \to W$ must satisfy $T(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2) = \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2)$. Cross product with a fixed vector is an example of a linear transformation.
- Given that the target space of a linear transformation T is $\mathbb{R}$, we say that T is a **linear functional**. Examples of linear functionals include (1.) the dot product with a fixed vector and (2.) the determinant.

True (a.) or False (b.)

The constant function $T(x, y, z) = 1$ is a linear functional.

True (a.) or False (b.)

The constant function $T(x, y, z) = 1$ is a linear functional.

(b.) False. On the contrary, if T were linear, we would have that $1 = T(2, 0, 0) = 2T(1, 0, 0) = 2$. Clearly, this is impossible.

True (a.) or False (b.)

The function $T(x, y, z) = (2x, 2y, 2z)$ is a linear transformation.

True (a.) or False (b.)

The function $T(x, y, z) = (2x, 2y, 2z)$ is a linear transformation.

(a.) True. Given vectors $\mathbf{v}_1 = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{v}_2 = \langle x_2, y_2, z_2 \rangle$ and scalars λ_1 and λ_2, we have that

$$\begin{aligned} T(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2) &= T(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2, \lambda_1 z_1 + \lambda_2 z_2) \\ &= (2\lambda_1 x_1 + 2\lambda_2 x_2, 2\lambda_1 y_1 + 2\lambda_2 y_2, 2\lambda_1 z_1 + 2\lambda_2 z_2) \\ &= (2\lambda_1 x_1, 2\lambda_1 y_1, 2\lambda_1 z_1) + (2\lambda_2 x_2, 2\lambda_2 y_2, 2\lambda_2 z_2) \\ &= \lambda_1(2x_1, 2y_1, 2z_1) + \lambda_2(2x_2, 2y_2, 2z_2) \\ &= \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2). \end{aligned}$$

True (a.) or False (b.)

The function $T(x, y, z) = (2x, 2y, 2z)$ is a linear transformation.

(a.) True. Given vectors $\mathbf{v}_1 = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{v}_2 = \langle x_2, y_2, z_2 \rangle$ and scalars λ_1 and λ_2, we have that

$$\begin{aligned} T(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2) &= T(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2, \lambda_1 z_1 + \lambda_2 z_2) \\ &= (2\lambda_1 x_1 + 2\lambda_2 x_2, 2\lambda_1 y_1 + 2\lambda_2 y_2, 2\lambda_1 z_1 + 2\lambda_2 z_2) \\ &= (2\lambda_1 x_1, 2\lambda_1 y_1, 2\lambda_1 z_1) + (2\lambda_2 x_2, 2\lambda_2 y_2, 2\lambda_2 z_2) \\ &= \lambda_1 (2x_1, 2y_1, 2z_1) + \lambda_2 (2x_2, 2y_2, 2z_2) \\ &= \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2). \end{aligned}$$

Essentially, one can think about this linear transformation as stretching a three-dimensional shape by a factor of two in each direction.