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- Given a function $f(x, y, z)$ such that the quantity

$$L = \lim_{||\mathcal{P}|| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{\ell} f(P_{ijk}) \Delta x \Delta y \Delta z$$

exists for all possible partitions P_{ijk} of the box $\mathcal{B} = [a, b] \times [c, d] \times [p, q]$, we say that $f(x, y, z)$ is (Riemann) **integrable** with triple integral

$$\iiint_{\mathcal{B}} f(x, y, z)\, dV = L.$$

Properties of Triple Integrals over Boxes

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- Like with double integrals, triple integrals are linear.
- Fubini's Theorem still applies to triple integrals, i.e., the triple integral of $f(x, y, z)$ on the box $\mathcal{B} = [a, b] \times [c, d] \times [p, q]$ is given by

$$\iiint_{\mathcal{B}} f(x, y, z)\, dV = \int_a^b \int_c^d \int_p^q f(x, y, z)\, dz\, dy\, dx.$$

Further, this iterated integral may be evaluated in $6 = 3 \cdot 2 \cdot 1$ ways.

Computing Double Integrals over Rectangles

True (a.) or False (b.)

The volume of the box $\mathcal{B} = [a, b] \times [c, d] \times [p, q]$ is given by

$$\int_p^q \int_c^d \int_a^b 1 \, dx \, dy \, dz.$$

True (a.) or False (b.)

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(a.) True. The triple integral evaluates to $(b - a)(d - c)(q - p)$, i.e., length $\times$ width $\times$ height. Of course, this is the volume of a box.

Triple Integration over General Regions

- If $\mathcal{W}$ is an x-simple region with projection $\mathcal{D}$ onto the yz-plane, i.e., $c \leq y \leq d$, $p \leq z \leq q$, and $g_1(y,z) \leq x \leq g_2(y,z)$, then

$$\iiint_{\mathcal{W}} f(x,y,z)\,dV = \iint_{\mathcal{D}} \left(\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\,dx \right) dA.$$

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- If $\mathcal{W}$ is a y-simple region with projection $\mathcal{D}$ onto the xz-plane, i.e., $a \leq x \leq b$, $p \leq z \leq q$, and $h_1(x,z) \leq y \leq h_2(x,z)$, then

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- If $\mathcal{W}$ is a z-simple region with projection $\mathcal{D}$ onto the xy-plane, i.e., $a \le x \le b$, $c \le y \le d$, and $k_1(x,y) \le z \le k_2(x,y)$, then

$$\iiint_{\mathcal{W}} f(x,y,z)\,dV = \iint_{\mathcal{D}} \left(\int_{k_1(x,y)}^{k_2(x,y)} f(x,y,z)\,dz \right) dA.$$

Converting to an Iterated Integral

Give three iterated integrals of the function $f(x, y, z) = xyz$ over the region $\mathcal{W}$ bounded by the equations $z = 4 - y^2$, $y = 2x$, $z = 0$, and $x = 0$ by projecting onto each of the coordinate planes.

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By drawing the graphs of each function in a Cartesian plane, we obtain numerical inequalities for each variable.

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By drawing the graphs of each function in a Cartesian plane, we obtain numerical inequalities for each variable. Consequently, we have that $0 \leq x \leq 1$, $0 \leq y \leq 2$, and $0 \leq z \leq 4$. We can now project $\mathcal{W}$ onto each of the three coordinate planes by manipulating these inequalities.

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By projecting onto the xy-plane, we can view $\mathcal{W}$ as a z-simple region.

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$$\iiint_{\mathcal{W}} xyz \, dV = \int_0^1 \int_{2x}^2 \int_0^{4-y^2} xyz \, dz \, dy \, dx.$$

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$$\iiint_{\mathcal{W}} xyz\, dV = \int_0^2 \int_0^{4-y^2} \int_0^{y/2} xyz\, dx\, dz\, dy.$$

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$$\iiint_{\mathcal{W}} xyz \, dV = \int_0^2 \int_0^{4-4x^2} \int_{2x}^{\sqrt{4-z}} xyz \, dy \, dz \, dx.$$