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- Using the Second Derivative Test, we were able to algorithmically generate data about the local extrema of a function $f(x, y)$.
- By the Extreme Value Theorem, we distinguished a nice class of functions and domains for which we could systematically describe the absolute extrema: **continuous** functions on **closed, bounded** domains achieve absolute extrema at critical or boundary points.

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- **Question:** Could we use calculus to maximize the utility (satisfaction) we derive from our meal at the buffet given that we can only eat so much chicken tikka masala and naan?

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- **Question:** Could we use calculus to maximize the utility (satisfaction) we derive from our meal at the buffet given that we can only eat so much chicken tikka masala and naan?
- Of course, the answer is yes, and we achieve this optimization via the method of Lagrange multipliers.

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- Consider walking along the curve $g(x,y)=k$ and stopping at the first point Q where $f(x,y)$ bumps into $g(x,y)=k$. We want to locate the point P at which $f(x,y)$ is a maximum. We note that ∇f_Q points in the direction of the maximum increase of f, but perhaps we cannot move straight along ∇f_Q because we will fall off the constraint curve; however, if we move along $g(x,y)=k$ in the direction of ∇f_Q, we will increase the value of $f(x,y)$. Continuing to move along the constraint curve in the direction of ∇f_Q eventually leads to a point P at which ∇f_P is orthogonal to $g(x,y)=k$. Consequently, $f(P)$ is a local maximum on $g(x,y)=k$. Considering that ∇g_P is orthogonal to the level curve $g(x,y)=k$ at P, we conclude that ∇f_P and ∇g_P are parallel so that $\nabla f_P = \lambda \cdot \nabla g_P$.

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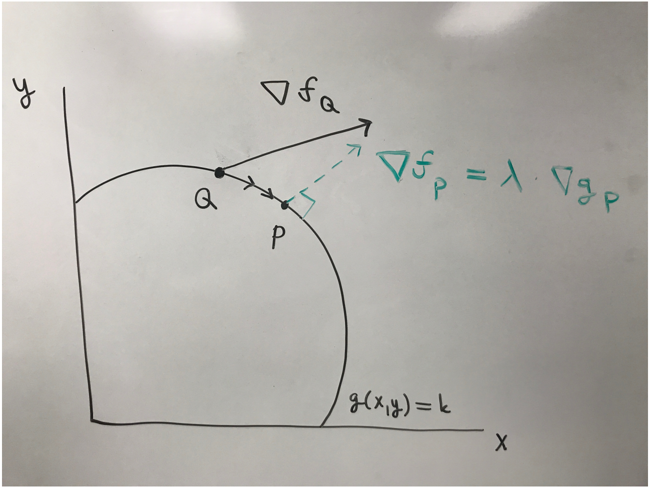

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Given differentiable functions $f(x, y)$ and $g(x, y)$ such that $f(x, y)$ has a local maximum (minimum) on the constraint curve $g(x, y) = k$ at a point $P = (a, b)$ and $\nabla g_P \neq (0, 0)$, there exists a nonzero scalar λ such that

$$\nabla f(a, b) = \lambda \cdot \nabla g(a, b),$$

i.e., $f_x(a, b) = \lambda \cdot g_x(a, b)$ and $f_y(a, b) = \lambda \cdot g_y(a, b)$.

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By the third equation, we have that $x = 24 - y$ so that $\lambda = 2(24 - y)y + y^2 = 48y - y^2$ by the first equation.

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True (a.) or False (b.)

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(b.) False. Consider the differentiable function $f(x, y) = x$ and the differentiable curve $g(x, y) = x - y = 0$. We note that all points on the line $y = x$ lie on the constraint curve; however, $f(x, y) = x$ can be arbitrarily large or small, hence it has no absolute extrema.

True (a.) or False (b.)

Given a differentiable function $f(x, y)$ and a differentiable curve $g(x, y) = k$, there exists a point P on the curve $g(x, y) = k$ such that $\nabla f_P = \lambda \cdot \nabla g_P$ for some real number λ.

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(b.) False. Consider the differentiable function $f(x, y) = x$ and the differentiable curve $g(x, y) = (x - 1)^3 - y^2 = 0$. We have that $\nabla f = (1, 0)$ and $\lambda \cdot \nabla g = (3(x - 1)^2, -2y)$ are not equal at any point P on the curve $g(x, y) = 0$.

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