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- We can make the notion of "(x, y) near (a, b)" precise, but essentially, a local maximum (minimum) is the largest (smallest) value a function takes in some region containing the point (a, b).

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- Back in Calculus I, we could detect the local extrema of a function $f(x)$ by checking the critical points of $f(x)$ against some test, i.e., by finding the values of x such that $f'(x) = 0$ or $f'(x)$ does not exist.

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- We say that a point $P = (a, b)$ is a **critical point** of a function $f(x, y)$ whenever we have that $\nabla f(P) = (f_x(P), f_y(P)) = (0, 0)$ or either of the first-order partial derivatives do not exist at P.

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Local Extrema

- We have just seen that critical points do not guarantee local extrema; however, the converse of this statement is true.

Local Extrema

Fermat's Theorem

Given that $f(x, y)$ has a local maximum (minimum) at a point $P = (a, b)$, it is guaranteed that P is a critical point of $f(x, y)$.

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Given that $f(x, y)$ has a local maximum (minimum) at a point $P = (a, b)$, it is guaranteed that P is a critical point of $f(x, y)$.

Put another way, if $P = (a, b)$ is not a critical point of $f(x, y)$, then $f(x, y)$ cannot have either a local maximum or a local minimum at P.

Local Extrema

True (a.) or False (b.)

$f(x, y) = x^2 + y^3 - 3y^2 - 9y$ has a critical point at $(0, -1)$.

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(a.) True. We have that

$$f_x = 2x \quad \text{and}$$
$$f_y = 3y^2 - 6y - 9 = 3(y^2 - 2y - 3) = 3(y+1)(y-3).$$

True (a.) or False (b.)

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Consequently, the critical points are $(0, 3)$ and $(0, -1)$.

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(b.) False. We have that $f_x = \frac{x}{|x|}$ and $f_y = \frac{y}{|y|}$. Evidently, these do not exist at the origin, hence the only critical point is $(0,0)$.

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- Given a function $f(x, y)$, we intuit that there should be some inequality involving $f_{xx}(a, b)$, $f_{yy}(a, b)$, $f_{xy}(a, b)$, and $f_{yx}(a, b)$ that determines if $P = (a, b)$ is a local extremum.

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- We define the **discriminant** $D(x, y)$ of the function $f(x, y)$ to be the determinant of the Hessian matrix of $f(x, y)$, i.e.,

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- Often, we deal with functions that satisfy Clairaut's Theorem, hence the discriminant can be written as

$$D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - f_{xy}^2(x, y).$$

The Second Derivative Test

Second Derivative Test

Consider a function $f(x, y)$ with a critical point $P = (a, b)$ such that the second-order partials f_{xx}, f_{yy}, f_{xy}, and f_{yx} exist and are continuous near P.

1. Given that $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, $f(a, b)$ is a local minimum.
2. Given that $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, $f(a, b)$ is a local maximum.
3. Given that $D(a, b) < 0$, P is a saddle point of $f(x, y)$.
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Given that $D(a, b) > 0$, $f_{xx}(a, b)f_{yy}(a, b) > 0$ so that $f_{xx}(a, b)$ and $f_{yy}(a, b)$ have the same sign. Consequently, the test can be run with $f_{yy}(a, b)$.

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We have that $f_x = 3x^2 - 12y$ and $f_y = 3y^2 - 12x$ so that the critical points occur when $x^2 = 4y$ and $x = \frac{1}{4}y^2$.

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$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = 0 \cdot 0 - (-12)^2 = -144 < 0 \quad \text{and}$$

$$D(4,4) = f_{xx}(4,4)f_{yy}(4,4) - f_{xy}^2(4,4) = (24)^2 - (-12)^2 > 0.$$

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We conclude that $(0, 0)$ is a saddle point and $(4, 4)$ is a local minimum.

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$$D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}^2(0, 0) = 0 \cdot 0 - (0)^2 = 0.$$

Consequently, the Second Derivative Test fails. Considering that $f(x, 0) = -x^3$ takes on both positive and negative values in a neighborhood of $(0, 0)$, we conclude that $(0, 0)$ is a saddle point.

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- Recall from Calculus I that a quadratic function $f(x) = a(x-h)^2 + k$ with $a > 0$ achieves its absolute minimum at its vertex (h, k); however, $f(x)$ has no absolute maximum over the real line $\mathbb{R}$.

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- Recall from Calculus I that a quadratic function $f(x) = a(x-h)^2 + k$ with $a > 0$ achieves its absolute minimum at its vertex (h, k); however, $f(x)$ has no absolute maximum over the real line $\mathbb{R}$.
- We can say even more about a function $f(x)$ and its absolute extrema.

Global Extrema

Extreme Value Theorem

Every continuous function $f(x)$ on a closed and bounded interval $[a, b]$ achieves its absolute maximum and its absolute minimum on $[a, b]$. Further, these extreme values occur either at the critical points $f'(x) = 0$ of $f(x)$ or at the end points of the interval $[a, b]$.

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- Our aim is therefore to characterize when a function of several variables exhibits any global extrema.

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True (a.) or False (b.)

$\mathcal{R} = \{(x, y) \mid x^2 + y^2 < 4\}$ is a closed region in $\mathbb{R}^2$.

True (a.) or False (b.)

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(b.) False. Geometrically, this is an open disk of radius 2.

True (a.) or False (b.)

$\mathcal{R} = \{(x, y) \mid x^2 + y^2 = 4\}$ is a closed region in $\mathbb{R}^2$.

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(a.) True.

True (a.) or False (b.)

$\mathcal{R} = \{(x, y) \,|\, x^2 + y^2 = 4\}$ is a closed region in $\mathbb{R}^2$.

(a.) True. Geometrically, this is a circle of radius 2.

True (a.) or False (b.)

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True (a.) or False (b.)

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(b.) False.

True (a.) or False (b.)

$\mathcal{R} = \{(x, y) \mid x^2 + y^2 > 4\}$ is a closed region in $\mathbb{R}^2$.

(b.) False. Geometrically, this is the Cartesian plane with a closed disk of radius 2 cut out. Complements of closed sets are open.

Extreme Value Theorem

- Understanding the properties of closed sets and recognizing them in practice is extremely useful in combination with the following fact.

Extreme Value Theorem

Extreme Value Theorem II

Every continuous function $f(x, y)$ on a closed and bounded region $\mathcal{R}$ in $\mathbb{R}^2$ achieves its absolute maximum and its absolute minimum on $\mathcal{R}$. Further, these extreme values occur either at the critical points $(f_x, f_y) = (0, 0)$ of $f(x, y)$ in the interior $\mathcal{R}^\circ$ of $\mathcal{R}$ or on the boundary $\partial\mathcal{R}$ of $\mathcal{R}$.

Extreme Value Theorem

Using the Extreme Value Theorem

Classify the extreme values of $f(x, y) = 2x - 3xy + y$ on the unit square $\mathcal{S} = \{(x, y) \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$.

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We have that $f_x = 2 - 3y$ and $f_y = 1 - 3x$ so that the only critical point occurs at $(\frac{1}{3}, \frac{2}{3})$.

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$$D\left(\frac{1}{3}, \frac{2}{3}\right) = f_{xx}\left(\frac{1}{3}, \frac{2}{3}\right) f_{yy}\left(\frac{1}{3}, \frac{2}{3}\right) - f_{xy}^2\left(\frac{1}{3}, \frac{2}{3}\right) = 0 \cdot 0 - (-3)^2 < 0.$$

Consequently, we have that $(\frac{1}{3}, \frac{2}{3})$ is a saddle point.

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Consequently, we have that $(\frac{1}{3}, \frac{2}{3})$ is a saddle point. Our global extrema must therefore occur on the boundary $\partial\mathcal{S}$. (Continued on the next slide.)

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edge	function	maximum	minimum
$y = 0$	$2x$	2	0
$y = 1$	$1 - x$	1	0
$x = 0$	y	1	0
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We conclude that $f(x, y)$ has an absolute maximum of 2 at $(1, 0)$ and an absolute minimum of 0 at both $(0, 0)$ and $(1, 1)$.

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We can systematically check the extreme values of f on the boundary $\partial\mathcal{D}^1$ by considering each arc of the disk separately.

1. On the arc $y = \sqrt{1 - x^2}$, we have the function $f(x) = x\sqrt{1 - x^2}$.

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2. On the arc $y = -\sqrt{1 - x^2}$, one can check that the critical values are the same as before with $y = -\frac{1}{\sqrt{2}}$.

Using the Extreme Value Theorem

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We had assumed previously that $x \neq \pm 1$ so that $f(x)$ was differentiable.

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Ultimately, we conclude that $f(x, y)$ has an absolute maximum value of $\frac{1}{2}$ occurring at both $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and an absolute minimum value of $-\frac{1}{2}$ occurring at both $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.