# The Chain Rule

- Back in Calculus I, we used the Chain Rule to compute derivatives of composite functions.

# The Chain Rule

- Back in Calculus I, we used the Chain Rule to compute derivatives of composite functions. Explicitly, the Chain Rule says that

$$\frac{d}{dx}g(f(x)) = g'(f(x))f'(x).$$

# The Chain Rule

- Back in Calculus I, we used the Chain Rule to compute derivatives of composite functions. Explicitly, the Chain Rule says that

$$\frac{d}{dx}g(f(x)) = g'(f(x))f'(x).$$

Primarily, the Chain Rule allows us to compute derivatives of functions that implicitly depend upon a variable $t$ — often time.

# The Chain Rule

- Back in Calculus I, we used the Chain Rule to compute derivatives of composite functions. Explicitly, the Chain Rule says that

$$\frac{d}{dx}g(f(x)) = g'(f(x))f'(x).$$

Primarily, the Chain Rule allows us to compute derivatives of functions that implicitly depend upon a variable $t$ — often time. Essentially, if $x = f(t)$ is a differentiable function of time, then $g(x) = g(f(t))$ is a differentiable function of time such that

$$\frac{d}{dt}g(x) = \frac{d}{dt}g(f(t)) = g'(f(t)) \cdot f'(t) = g'(x) \cdot f'(t) = \frac{dg}{dx}\frac{dx}{dt}.$$

# The Chain Rule

- Of course, we can generalize this idea to functions of several variables, each of which depends implicitly on another variable $t$.

# The Chain Rule

- Of course, we can generalize this idea to functions of several variables, each of which depends implicitly on another variable $t$. Explicitly, given differentiable functions $x = g(t)$ and $y = h(t)$, $z = f(x, y) = f(x(t), y(t))$ is a differentiable function of $t$ with

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

# The Chain Rule

- Of course, we can generalize this idea to functions of several variables, each of which depends implicitly on another variable $t$. Explicitly, given differentiable functions $x = g(t)$ and $y = h(t)$, $z = f(x, y) = f(x(t), y(t))$ is a differentiable function of $t$ with

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

- We note that this formula is specific neither to the number of variables $x_1, \ldots, x_n$ upon which the function $f$ depends nor the number of implicit variables $t_1, \ldots, t_k$ upon which those $x_i$ variables depend.

# The Chain Rule

- Of course, we can generalize this idea to functions of several variables, each of which depends implicitly on another variable $t$. Explicitly, given differentiable functions $x = g(t)$ and $y = h(t)$, $z = f(x, y) = f(x(t), y(t))$ is a differentiable function of $t$ with

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

- We note that this formula is specific neither to the number of variables $x_1, \ldots, x_n$ upon which the function $f$ depends nor the number of implicit variables $t_1, \ldots, t_k$ upon which those $x_i$ variables depend. Ultimately, the recipe is always given by

$$\frac{\partial f}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t_j} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_j}.$$

# The Chain Rule

- Consider the function $f(x, y)$. We can take the partial derivatives of $f$ in terms of polar coordinates $(r, \theta)$ since $x = r\cos\theta$ and $y = r\sin\theta$ are both differentiable functions of $r$ and $\theta$.

# The Chain Rule

- Consider the function $f(x, y)$. We can take the partial derivatives of $f$ in terms of polar coordinates $(r, \theta)$ since $x = r\cos\theta$ and $y = r\sin\theta$ are both differentiable functions of $r$ and $\theta$. Explicitly, we compute

$$\frac{\partial x}{\partial r} = \cos\theta,$$

# The Chain Rule

- Consider the function $f(x, y)$. We can take the partial derivatives of $f$ in terms of polar coordinates $(r, \theta)$ since $x = r\cos\theta$ and $y = r\sin\theta$ are both differentiable functions of $r$ and $\theta$. Explicitly, we compute

$$\frac{\partial x}{\partial r} = \cos\theta, \ \frac{\partial x}{\partial \theta} = -r\sin\theta,$$

- Consider the function $f(x, y)$. We can take the partial derivatives of $f$ in terms of polar coordinates $(r, \theta)$ since $x = r\cos\theta$ and $y = r\sin\theta$ are both differentiable functions of $r$ and $\theta$. Explicitly, we compute

$$\frac{\partial x}{\partial r} = \cos\theta,\ \ \frac{\partial x}{\partial \theta} = -r\sin\theta,\ \ \frac{\partial y}{\partial r} = \sin\theta,$$

# The Chain Rule

- Consider the function $f(x, y)$. We can take the partial derivatives of $f$ in terms of polar coordinates $(r, \theta)$ since $x = r\cos\theta$ and $y = r\sin\theta$ are both differentiable functions of $r$ and $\theta$. Explicitly, we compute

$$\frac{\partial x}{\partial r} = \cos\theta, \ \frac{\partial x}{\partial \theta} = -r\sin\theta, \ \frac{\partial y}{\partial r} = \sin\theta, \ \frac{\partial y}{\partial \theta} = r\cos\theta,$$

# The Chain Rule

- Consider the function $f(x, y)$. We can take the partial derivatives of $f$ in terms of polar coordinates $(r, \theta)$ since $x = r\cos\theta$ and $y = r\sin\theta$ are both differentiable functions of $r$ and $\theta$. Explicitly, we compute

$$\frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial x}{\partial \theta} = -r\sin\theta,\ \frac{\partial y}{\partial r} = \sin\theta,\ \frac{\partial y}{\partial \theta} = r\cos\theta,$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta = \frac{xf_x + yf_y}{\sqrt{x^2 + y^2}} \qquad \text{and}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta f_x + r\cos\theta f_y = -yf_x + xf_y.$$

# The Chain Rule

- Computing derivatives of multivariable functions by the Chain Rule can be reduced to a relatively simple algorithm. For simplicity, we will demonstrate the algorithm on $f(x, y) = f(x(s, t), y(s, t))$.

- Computing derivatives of multivariable functions by the Chain Rule can be reduced to a relatively simple algorithm. For simplicity, we will demonstrate the algorithm on $f(x, y) = f(x(s, t), y(s, t))$.

  1. Compute the partials of $f$ with respect to $x$ and $y$.

# The Chain Rule

- Computing derivatives of multivariable functions by the Chain Rule can be reduced to a relatively simple algorithm. For simplicity, we will demonstrate the algorithm on $f(x, y) = f(x(s, t), y(s, t))$.

  1. Compute the partials of $f$ with respect to $x$ and $y$.
  2. Compute the partials of $x$ and $y$ with respect to $s$.

# The Chain Rule

- Computing derivatives of multivariable functions by the Chain Rule can be reduced to a relatively simple algorithm. For simplicity, we will demonstrate the algorithm on $f(x, y) = f(x(s, t), y(s, t))$.

  1. Compute the partials of $f$ with respect to $x$ and $y$.
  2. Compute the partials of $x$ and $y$ with respect to $s$.
  3. By the Chain Rule, we have that $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$.

# The Chain Rule

- Computing derivatives of multivariable functions by the Chain Rule can be reduced to a relatively simple algorithm. For simplicity, we will demonstrate the algorithm on $f(x, y) = f(x(s, t), y(s, t))$.

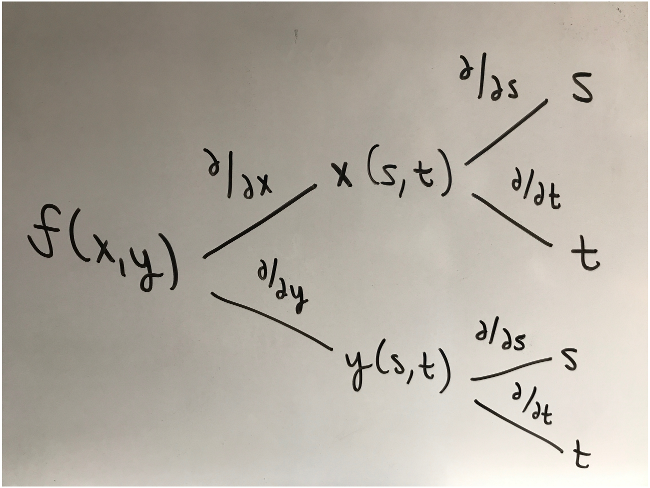
1. Compute the partials of $f$ with respect to $x$ and $y$.
2. Compute the partials of $x$ and $y$ with respect to $s$.
3. By the Chain Rule, we have that $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$.
4. Compute the partials of $x$ and $y$ with respect to $t$.

# The Chain Rule

- Computing derivatives of multivariable functions by the Chain Rule can be reduced to a relatively simple algorithm. For simplicity, we will demonstrate the algorithm on $f(x, y) = f(x(s, t), y(s, t))$.

  1. Compute the partials of $f$ with respect to $x$ and $y$.
  2. Compute the partials of $x$ and $y$ with respect to $s$.
  3. By the Chain Rule, we have that $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$.
  4. Compute the partials of $x$ and $y$ with respect to $t$.
  5. By the Chain Rule, we have that $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}$.

# The Chain Rule



# The Chain Rule

- Given that $x(s,t) = t^2$, $y(s,t) = st$, and $z(s,t) = t - s$, let us use the Chain Rule to compute $\frac{\partial f}{\partial t}$ of $f(x,y,z) = e^{xyz}$.

# The Chain Rule

- Given that $x(s,t) = t^2$, $y(s,t) = st$, and $z(s,t) = t - s$, let us use the Chain Rule to compute $\frac{\partial f}{\partial t}$ of $f(x,y,z) = e^{xyz}$.

  1. Compute the partials of $f$ with respect to $x$, $y$, and $z$.

# The Chain Rule

- Given that $x(s,t) = t^2$, $y(s,t) = st$, and $z(s,t) = t - s$, let us use the Chain Rule to compute $\frac{\partial f}{\partial t}$ of $f(x,y,z) = e^{xyz}$.

  1. Compute the partials of $f$ with respect to $x$, $y$, and $z$.

$$\frac{\partial f}{\partial x} = yze^{xyz}, \ \frac{\partial f}{\partial y} = xze^{xyz}, \ \frac{\partial f}{\partial z} = xye^{xyz}$$

# The Chain Rule

- Given that $x(s,t) = t^2$, $y(s,t) = st$, and $z(s,t) = t - s$, let us use the Chain Rule to compute $\frac{\partial f}{\partial t}$ of $f(x,y,z) = e^{xyz}$.

1. Compute the partials of $f$ with respect to $x$, $y$, and $z$.

$$\frac{\partial f}{\partial x} = yze^{xyz}, \ \frac{\partial f}{\partial y} = xze^{xyz}, \ \frac{\partial f}{\partial z} = xye^{xyz}$$

2. Compute the partials of $x$, $y$, and $z$ with respect to $t$.

- Given that $x(s,t) = t^2$, $y(s,t) = st$, and $z(s,t) = t - s$, let us use the Chain Rule to compute $\frac{\partial f}{\partial t}$ of $f(x,y,z) = e^{xyz}$.

1. Compute the partials of $f$ with respect to $x$, $y$, and $z$.

$$\frac{\partial f}{\partial x} = yze^{xyz}, \ \frac{\partial f}{\partial y} = xze^{xyz}, \ \frac{\partial f}{\partial z} = xye^{xyz}$$

2. Compute the partials of $x$, $y$, and $z$ with respect to $t$.

$$\frac{\partial x}{\partial t} = 2t, \ \frac{\partial y}{\partial t} = s, \ \frac{\partial z}{\partial t} = 1$$

# The Chain Rule

- Given that $x(s,t) = t^2$, $y(s,t) = st$, and $z(s,t) = t - s$, let us use the Chain Rule to compute $\frac{\partial f}{\partial t}$ of $f(x,y,z) = e^{xyz}$.

  1. Compute the partials of $f$ with respect to $x$, $y$, and $z$.

  $$\frac{\partial f}{\partial x} = yze^{xyz}, \ \frac{\partial f}{\partial y} = xze^{xyz}, \ \frac{\partial f}{\partial z} = xye^{xyz}$$

  2. Compute the partials of $x$, $y$, and $z$ with respect to $t$.

  $$\frac{\partial x}{\partial t} = 2t, \ \frac{\partial y}{\partial t} = s, \ \frac{\partial z}{\partial t} = 1$$

  3. By the Chain Rule, we have that $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$.

# The Chain Rule

- Given that $x(s,t) = t^2$, $y(s,t) = st$, and $z(s,t) = t - s$, let us use the Chain Rule to compute $\frac{\partial f}{\partial t}$ of $f(x,y,z) = e^{xyz}$.

1. Compute the partials of $f$ with respect to $x$, $y$, and $z$.

$$\frac{\partial f}{\partial x} = yze^{xyz}, \ \frac{\partial f}{\partial y} = xze^{xyz}, \ \frac{\partial f}{\partial z} = xye^{xyz}$$

2. Compute the partials of $x$, $y$, and $z$ with respect to $t$.

$$\frac{\partial x}{\partial t} = 2t, \ \frac{\partial y}{\partial t} = s, \ \frac{\partial z}{\partial t} = 1$$

3. By the Chain Rule, we have that $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$.

$$\frac{\partial f}{\partial t} = e^{t^3 s(t-s)}[(st)(t-s)(2t) + (t^2)(t-s)(s) + (t^2)(st)(1)]$$

# The Chain Rule

- Given differentiable functions $x = h(s,t)$, $y = k(s,t)$, $w = f(x,y)$, and $z = g(x,y)$, use the table to compute the given derivatives.

$$\frac{\partial w}{\partial x} = 2 \qquad \frac{\partial z}{\partial x} = \quad 3 \qquad \frac{\partial x}{\partial s} = -1 \qquad \frac{\partial x}{\partial t} = \quad 1$$

$$\frac{\partial w}{\partial y} = -3 \qquad \frac{\partial z}{\partial y} = 2 \qquad \frac{\partial y}{\partial s} = -2 \qquad \frac{\partial y}{\partial t} = -1$$

1. Compute the value of $\frac{\partial}{\partial s}(w - z)$ whenever $w = 10$ and $z = -7$.

$$\frac{\partial}{\partial s}(w - z) = \frac{\partial w}{\partial s} - \frac{\partial z}{\partial s}$$

$$= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} - \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} - \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= (2)(-1) + (-3)(-2) - (3)(-1) - (2)(-2)$$

$$= -2 + 6 + 3 + 4 = 11$$

# The Chain Rule

2. Compute the value of $\frac{\partial}{\partial t}\left(\frac{\tan z}{w}\right)$ whenever $w = 1$ and $z = \frac{\pi}{3}$.

$$\frac{\partial}{\partial t}\left(\frac{\tan z}{w}\right) = \frac{w \cdot \sec^2 z \cdot \frac{\partial z}{\partial t} - \tan z \cdot \frac{\partial w}{\partial t}}{w^2}$$

$$= (2)^2\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right) - \sqrt{3}\left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}\right)$$

$$= 4[(3)(1) + (2)(-1)] - \sqrt{3}[(2)(1) + (-3)(-1)] = 4 - 5\sqrt{3}$$