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  - $\nabla(g \circ f) = (g' \circ f)\nabla f$ for any differentiable function g of one variable, i.e., the Chain Rule holds.

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Once all the dust settles in the limit, we have that

$$D_{\mathbf{u}} f(P) = \nabla f_P \cdot \mathbf{u}.$$

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$$D_{\mathbf{u}} f(P) = ||\nabla f_P|| \cos\theta,$$

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where ψ is the angle of inclination. By our previous discussion, the steepest direction on the surface $z = f(x, y)$ is toward ∇f_P.

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By walking in the direction of $\frac{1}{||\nabla f_P||} \nabla f_P$, we would encounter the steepest slopes, and the angle of inclination would be

$$\psi = \tan^{-1}(||\nabla f_P||) = \tan^{-1}(\sqrt{2^2 + 3^2}) \approx 74.5^\circ.$$