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- Basically, a function is locally linear whenever its graph is a line when magnified sufficiently many times under a microscope.
- Explicitly, a function $f(x)$ is locally linear at $x = a$ whenever

$$\lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} = 0,$$

i.e., $\lim_{x \to a} f'(x) = f'(a)$, i.e., $f'(x)$ is continuous at $x = a$.

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i.e., $\lim_{x \to a} f'(x) = f'(a)$, i.e., $f'(x)$ is continuous at $x = a$. We will return to this notion for functions of several variables

The Tangent Plane

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- Given a function $f(x, y)$ with partial derivatives $f_x(a, b)$ and $f_y(a, b)$ at (a, b), we note that the vector $\mathbf{u} = \langle 1, 0, f_x(a, b)\rangle$ determines the line tangent to $f(a, b)$ in the x-direction and $\mathbf{v} = \langle 0, 1, f_y(a, b)\rangle$ determines the line tangent to $f(a, b)$ in the y-direction.

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- Consequently, we may take $\mathbf{n} = \mathbf{v} \times \mathbf{u} = \langle f_x(a, b), f_y(a, b), -1 \rangle$ so that

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$$

is the equation of the plane tangent to $f(x, y)$ at (a, b).

Locally Linear Property

- Using the equation of the tangent plane as our guide, we define the **linearization** of $f(x, y)$ at (a, b) to be the linear function

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

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- We say that $f(x, y)$ is **locally linear** at (a, b) whenever

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i.e., the vertical distance from $f(x, y)$ to $L(x, y)$ tends to zero faster than the distance from (x, y) to (a, b) as (x, y) tends to (a, b).

The Tangent Plane

Equation of the Tangent Plane

Given a function $f(x, y)$ that is locally linear at (a, b), the equation of the plane tangent to $f(x, y)$ at the point (a, b) can be written as

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

Observe that this is a generalization of the tangent line $y = f'(a)(x - a)$.

The Tangent Plane

True (a.) or False (b.)

The tangent plane approximation of the function $f(x, y) = x^2 + y^2$ at the origin $(0, 0)$ is given by the equation $z = 0$.

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The tangent plane approximation of the function $f(x, y) = x^2 + y^2$ at the origin $(0, 0)$ is given by the equation $z = 0$.

(a.) True. We have that $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$ so that $f_x(0, 0) = f_y(0, 0) = f(0, 0)$.

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One other way to see this is that the graph of $f(x, y) = x^2 + y^2$ is an elliptic paraboloid with an absolute minimum at $(0, 0)$.

Differentiability

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- Of course, in practice, this condition would be tedious to check.

Differentiability

Criteria for Differentiability

Given that the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ both exist and are continuous (as functions) near (a, b), $f(x, y)$ is differentiable at (a, b).

True (a.) or False (b.)

$f(x, y) = \sqrt{x^2 + y^2}$ is differentiable on $\mathbb{R}^2 - \{(0, 0)\}$.

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$f(x,y) = \sqrt{x^2+y^2}$ is differentiable on $\mathbb{R}^2 - \{(0,0)\}$.

(a.) True. We have that

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \text{ and } f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}},$$

and these exist and are continuous on $\mathbb{R}^2 - \{(0,0)\}$.

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and these exist and are continuous on $\mathbb{R}^2 - \{(0,0)\}$. We claim further that neither f_x nor f_y exist at the origin. Considering that $f(x,0) = |x|$ and $f(0,y) = |y|$ are not differentiable at the origin, it follows that the limits $f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x}$ and $f_y(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y}$ DNE.

Differentials and Linear Approximation

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$$f(x, y) \approx f_x(a, b)(x - a) + f_y(a, b)(y - a) + f(a, b).$$

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$$f(x, y) \approx f_x(a, b)(x - a) + f_y(a, b)(y - a) + f(a, b).$$

Often, we will use the notation $\Delta f = f(x, y) - f(a, b)$, $\Delta x = x - a$, and $\Delta y = y - b$ so that the tangent plane approximation becomes

$$\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y.$$

Using a Linear Approximation

Compute the value of $\sqrt{3.01^2 + 3.99^2}$ using a linear approximation.

(a.) 5.000

(b.) 4.998

(c.) 5.002

(d.) 7.000

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Using the function $f(x, y) = \sqrt{x^2 + y^2}$, we compute the tangent plane approximation at the point $(3.01, 3.99)$ with $a = 3$ and $b = 4$. We have that $f_x(3, 4) = 0.6$, $f_y(3, 4) = 0.8$, and $f(3, 4) = 5$ so that

$$f(3.01, 3.99) \approx 0.6(3.01 - 3) + 0.8(3.99 - 4) + 5 = 4.998.$$