Partial Derivatives

- Recall that the derivative $f'(x)$ of a function $f(x)$ of one variable is given by the limit of the difference quotient of $f(x)$ on the interval $[x, x+h]$ as h approaches 0. Explicitly, we have that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

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- Considering that this is a function of x, we may compute the derivative $f'(a)$ of $f(x)$ at the point $x = a$ by evaluating the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Partial Derivatives

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$$\frac{\partial}{\partial x} f(a, b) \overset{\text{def}}{=} f_x(a, b) \overset{\text{def}}{=} \frac{d}{dx} g(x) \bigg|_{x=a} = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

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- Likewise, we may perform a similar process with y. We refer to the resulting quantity $\frac{\partial}{\partial y} f(a, b) = f_y(a, b)$ as the partial derivative of $f(x, y)$ with respect to y at the point (a, b).

True (a.) or False (b.)

Given that $f(x, y) = x^2 + y^2$, evaluating the limit $\lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h}$ gives the quantity $f_x(2, 3)$.

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(a.) True. By definition, we have that

$$f_x(2,3) = \lim_{h\to 0} \frac{f(2+h,3) - f(2,3)}{h} = \lim_{h\to 0} \frac{(2+h)^2 + 3^2 - (2^2+3^2)}{h}.$$

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Cancelling both 3^2 terms gives us the limit in question.

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(a.) True. By definition, we have that

$$\begin{aligned} f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{r(x) + s(y+h) - (r(x) + s(y))}{h} \\ &= \lim_{h\to 0} \frac{s(y+h) - s(y)}{h}, \end{aligned}$$

and this last expression is exactly $\frac{d}{dy}s(y)$.

Computing Partial Derivatives

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- Explicitly, given a function $f(x, y)$, if we wish to compute $f_x(x, y)$, we may substitute a box $\square$ in place of the variable y; subsequently compute the derivative of $g(x) = f(x, \square)$ by treating the boxes as constants; and finally replace all of the boxes with the variable y.

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- Likewise, we may compute $f_y(x, y)$ by a similar process that replaces each occurrence of x with a diamond $\Diamond$.

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$$f'(x) = \frac{1}{1 + \frac{\square^2}{x^2}} \cdot \left(- \frac{\square}{x^2} \right) = - \frac{\square}{x^2 + \square^2}$$

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  By a similar process using x and $\diamondsuit$, we have that $f_y(x, y) = \dfrac{x}{x^2 + y^2}$.

Using the Chain Rule

For which of the following functions must one implement the Chain Rule when taking the partial derivative with respect to x?

(a.) $f(x, y) = \dfrac{xy}{\sin x}$

(b.) $g(x, y) = \arctan\left(\frac{x}{y}\right)$

(c.) $h(x, y) = x^2 y^3$

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We note that the partial derivative of $\frac{x}{y}$ with respect to x is $\frac{1}{y}$, hence we must apply the Chain Rule to compute this derivative.

Using the Product Rule

For which of the following functions must one implement the Product Rule when taking the partial derivative with respect to z?

(a.) $f(x, y, z) = x + y + z$

(b.) $g(x, y, z) = e^{x^{y^z}}$

(c.) $h(x, y, z) = xy + \cos(z^2)$

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We have a product of functions that both involve z, hence we must use the Product Rule with respect to z to compute this derivative.

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- Like with derivatives of functions of one variable, we can take higher order partial derivatives of functions of several variables; however, we now have a choice as to which variable we differentiate.

- Consider the function $f(x, y) = x^2 + y^2$. We have that $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, but each of these functions is differentiable with respect to both x and y, hence we can compute

$$f_{xx}(x, y) = \frac{\partial}{\partial x} f_x(x, y) = 2 = \frac{\partial}{\partial y} f_y(x, y) = f_{yy}(x, y),$$

and we can also compute the **mixed partials**

$$f_{xy}(x, y) = \frac{\partial}{\partial y} f_x(x, y) = 0 = \frac{\partial}{\partial x} f_y(x, y) = f_{yx}(x, y).$$

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Observe that $f_{xx} = f_{yy}$ and $f_{yx} = f_{xy}$. One is coincident; one is not.

Higher Order Partial Derivatives

Clairaut's Theorem

Given a function $f(x_1, \ldots, x_n)$ whose kth-order partial derivatives are all continuous, the kth mixed partials of f can be computed in any order.

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Given a function $f(x_1, \ldots, x_n)$ whose kth-order partial derivatives are all continuous, the kth mixed partials of f can be computed in any order.

Clairaut's Theorem asserts that the equality of f_{yx} and f_{xy} in the previous example is not a coincidence but that in fact this is always the case so long as the second-order partial derivatives are all continuous.

True (a.) or False (b.)

Given a function $f(x, y)$, the mixed partial derivative f_{xyxy} can be found by first taking the partial derivative in x, then taking the partial derivative in y, then in x again, and finally in y again.

True (a.) or False (b.)

Given a function $f(x, y)$, the mixed partial derivative f_{xyxy} can be found by first taking the partial derivative in x, then taking the partial derivative in y, then in x again, and finally in y again.

(b.) True.

Higher Order Partial Derivatives

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(b.) True. We read mixed partial derivatives in subscript notation from left to right.

Higher Order Partial Derivatives

True (a.) or False (b.)

Given a function $f(x, y)$, the mixed partial derivative f_{xyxy} can be found by applying the differential operator $\frac{\partial^4}{\partial x \partial y \partial x \partial y}$ to $f(x, y)$.

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(b.) False. We read mixed partial derivatives in operator notation from right to left, hence we have that $\frac{\partial^4}{\partial x \partial y \partial x \partial y} f(x, y) = f_{yxyx}$.

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(b.) False. We read mixed partial derivatives in operator notation from right to left, hence we have that $\frac{\partial^4}{\partial x \partial y \partial x \partial y} f(x, y) = f_{yxyx}$. Unless Clairaut's Theorem holds for $f(x, y)$, this function might not be the same as f_{xyxy}.

Using Clairaut's Theorem

Given that $f(w, x, y, z) = 2^{xy} + 3^{xz} + 5^{yz} + w^7$, compute f_{zyxw}.

(a.) $f_{zyxw} = -\frac{5 \ln y}{z^2}$

(b.) $f_{zyxw} = 7 \cdot 6 \cdot 5 \cdot 4$

(c.) $f_{zyxw} = 0$

(d.) $f_{zyxw} = 2x + 3z + 5y + 7$

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We note that the fourth-order mixed partial derivatives are all continuous since they are sums and products of compositions of continuous functions. By Clairaut's Theorem, we have that $f_{zyxw} = f_{wxyz} = \frac{\partial^3}{\partial z \partial y \partial x} 7w^6 = 0$.