Limits in Several Variables

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- Likewise, the limit $\lim_{(x,y) \to (a,b)} f(x, y)$ of a function $f(x, y)$ as (x, y) approaches the point (a, b) measures the behavior of $f(x, y)$ when the distance $\sqrt{(x-a)^2 + (y-b)^2}$ from (x, y) to (a, b) is very small.

Criteria for Existence of Limits in Several Variables

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- **Caution:** If the limit fails to exist for any path from (x, y) to (a, b), then the limit does not exist.

- **Caution:** If the limit obtained from one path does not equal the limit obtained from another path, then the limit does not exist.

True (a.) or False (b.)

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One other way to see it is to convert to polar coordinates. We have that $x = r\cos\theta$ so that by the constant rule for limits, we conclude that

$$\lim_{(x,y)\to(0,0)} x = \lim_{r\to 0}(r\cos\theta) = \cos\theta \cdot (\lim_{r\to 0} r) = 0.$$

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Consequently, for different choices of $m \neq 0$, we obtain a different limit.

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    4. $\lim \left(\dfrac{f}{g}\right) = \dfrac{\lim f}{\lim g}$ whenever $\lim g \neq 0$

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- Composition of continuous functions preserves continuity.
- Polynomials, rational functions, power functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions, and any composite of these are continuous.

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(a.) True. Compositions of continuous functions are continuous. Given that $\frac{1}{f(x,y)}$ is continuous at P, we have that $f(x, y)$ is continuous at P since $\frac{1}{x}$ is continuous for $x \neq 0$ and $f(x, y) = \frac{1}{x} \circ \frac{1}{f(x,y)}$.

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- **Strategy:** Given that all else has failed, convert to polar or cylindrical coordinates. If the limit depends on the angle θ, then it does not exist; otherwise, the limit exists and equals the computed number.

Computing Limits in Several Variables

True (a.) or False (b.)

Given that $f(x,0) = 3$ for $x \neq 0$ and $f(0,y) = 5$ for $y \neq 0$, we have that $\lim_{(x,y)\to(0,0)} f(x,y) = (3,5)$.

True (a.) or False (b.)

Given that $f(x,0)=3$ for $x\neq 0$ and $f(0,y)=5$ for $y\neq 0$, we have that $\lim_{(x,y)\to(0,0)} f(x,y)=(3,5)$.

(a.) False. Consider the piecewise function

$$f(x,y)=\begin{cases}0, & \text{if } x=0, y=0;\\ 3, & \text{if } x\neq 0, y=0;\\ 5 & \text{if } x=0, y\neq 0; \text{ and}\\ \frac{1}{x+y} & \text{if } x\neq 0, y\neq 0.\end{cases}$$

On the path $y=x$, $\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{x\to 0}\frac{1}{2x}$ does not exist.

Computing Limits in Two Variables

Compute the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2+y^2}$.

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(b.) DNE

(c.) $\frac{1}{3}$

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We first plug and chug to find the indeterminate form $\frac{0}{0}$. We try the x- and y-axes to find that the limits both exist and equal 0. Using the path $y = mx$, however, yields $\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{(3+m^2)x^2} = \frac{m}{3+m^2}$.