Functions of Several Variables

- Like functions of a single variable, a function $f : \mathbb{R}^n \to \mathbb{R}$ in n variables is a rule that assigns to each point $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ one and only one value $f(x_1, \ldots, x_n)$ in $\mathbb{R}$.

Functions of Several Variables

- Like functions of a single variable, a function $f : \mathbb{R}^n \to \mathbb{R}$ in n variables is a rule that assigns to each point $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ one and only one value $f(x_1, \ldots, x_n)$ in $\mathbb{R}$.
- For instance, we have the function $f(x, y) = x^2 + y^2$.

Functions of Several Variables

- Like functions of a single variable, a function $f : \mathbb{R}^n \to \mathbb{R}$ in n variables is a rule that assigns to each point $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ one and only one value $f(x_1, \ldots, x_n)$ in $\mathbb{R}$.
- For instance, we have the function $f(x, y) = x^2 + y^2$. Observe that the image of $\mathbb{R}^2$ under f is the paraboloid $z = x^2 + y^2$.

Functions of Several Variables

- Like functions of a single variable, a function $f : \mathbb{R}^n \to \mathbb{R}$ in n variables is a rule that assigns to each point $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ one and only one value $f(x_1, \ldots, x_n)$ in $\mathbb{R}$.
- For instance, we have the function $f(x, y) = x^2 + y^2$. Observe that the image of $\mathbb{R}^2$ under f is the paraboloid $z = x^2 + y^2$.
- Even more complicated, $f(x, y, z) = 3x - y^3 + e^z$ is also a function.

Functions of Several Variables

- Like functions of a single variable, a function $f : \mathbb{R}^n \to \mathbb{R}$ in n variables is a rule that assigns to each point $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ one and only one value $f(x_1, \ldots, x_n)$ in $\mathbb{R}$.
- For instance, we have the function $f(x, y) = x^2 + y^2$. Observe that the image of $\mathbb{R}^2$ under f is the paraboloid $z = x^2 + y^2$.
- Even more complicated, $f(x, y, z) = 3x - y^3 + e^z$ is also a function. Unfortunately, the image of $\mathbb{R}^3$ under f is a four-dimensional object, hence we cannot picture it as some familiar geometric shape.

Functions of Several Variables

- We say that the **domain** of a function is the set

$$D_f = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid f(x_1, \ldots, x_n) \in \mathbb{R}\}$$

of points for which the rule makes sense.

- We say that the **domain** of a function is the set

$$D_f = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid f(x_1, \ldots, x_n) \in \mathbb{R}\}$$

  of points for which the rule makes sense.

- Generally, the domain of a function is not $\mathbb{R}^n$. For instance, we cannot divide by 0, and we cannot take square roots of negative numbers.

Functions of Several Variables

- We say that the **domain** of a function is the set

$$D_f = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, f(x_1, \ldots, x_n) \in \mathbb{R}\}$$

  of points for which the rule makes sense.

- Generally, the domain of a function is not $\mathbb{R}^n$. For instance, we cannot divide by 0, and we cannot take square roots of negative numbers.

- We say that the **range** (or **codomain**) of a function is the set

$$R_f = \{f(x_1, \ldots, x_n) \in \mathbb{R} \,|\, (x_1, \ldots, x_n) \in D_f\}$$

  of outputs given by a function for all possible inputs.

Domain and Range

Give the domain of the function $f(x, y) = \sqrt{-x^2 + 16 + y}$.

(a.) $x \geq 0$ and $y \geq 0$

(b.) $y \geq x^2 - 16$

(c.) $-4 \leq x \leq 4$ and $y \geq 0$

(d.) $\mathbb{R}^2$

Domain and Range

Give the domain of the function $f(x, y) = \sqrt{-x^2 + 16 + y}$.

(a.) $x \geq 0$ and $y \geq 0$

(b.) $y \geq x^2 - 16$

(c.) $-4 \leq x \leq 4$ and $y \geq 0$

(d.) $\mathbb{R}^2$

We must have that $-x^2 + 16 + y \geq 0$ so that $y \geq x^2 - 16$.

Domain and Range

Give the range of the function $f(x, y) = \sqrt{-x^2 + 16 + y}$.

(a.) $f(x, y) \geq 0$

(b.) $f(x, y) \neq 0$

(c.) $\mathbb{R}$

(d.) $\mathbb{R}^2$

Domain and Range

Give the range of the function $f(x,y) = \sqrt{-x^2 + 16 + y}$.

(a.) $f(x,y) \geq 0$

(b.) $f(x,y) \neq 0$

(c.) $\mathbb{R}$

(d.) $\mathbb{R}^2$

Every non-negative real number can be obtained from this function. Explicitly, we have that $a = \sqrt{a^2} = \sqrt{-4^2 + 16 + a^2}$ for every $a > 0$.

Level Curves and Traces of Functions

- Like we did with quadric surfaces, we can identify a function in several variables by looking at its **traces**, i.e., its intersections with planes of the form $x_i = C$ for some real number C.

Level Curves and Traces of Functions

- Like we did with quadric surfaces, we can identify a function in several variables by looking at its **traces**, i.e., its intersections with planes of the form $x_i = C$ for some real number C.
- For instance, the vertical traces of the function $f(x, y) = x \sin y$ are the sine curves $f(C, y) = C \sin y$ of amplitude C in the plane $x = C$ and the lines $f(x, C) = x \sin C$ of slope $\sin C$ in the plane $y = C$.

Level Curves and Traces of Functions

- Like we did with quadric surfaces, we can identify a function in several variables by looking at its **traces**, i.e., its intersections with planes of the form $x_i = C$ for some real number C.
- For instance, the vertical traces of the function $f(x, y) = x \sin y$ are the sine curves $f(C, y) = C \sin y$ of amplitude C in the plane $x = C$ and the lines $f(x, C) = x \sin C$ of slope $\sin C$ in the plane $y = C$.
- **Level curves** are obtain by taking the horizontal traces of a function of several variables and projecting them into the xy-plane.

Level Curves and Traces of Functions

- Like we did with quadric surfaces, we can identify a function in several variables by looking at its **traces**, i.e., its intersections with planes of the form $x_i = C$ for some real number C.

- For instance, the vertical traces of the function $f(x, y) = x \sin y$ are the sine curves $f(C, y) = C \sin y$ of amplitude C in the plane $x = C$ and the lines $f(x, C) = x \sin C$ of slope $\sin C$ in the plane $y = C$.

- **Level curves** are obtain by taking the horizontal traces of a function of several variables and projecting them into the xy-plane.

- We have already seen that the level curves of the elliptic paraboloid $f(x, y) = x^2 + 3y^2$ are ellipses;

Level Curves and Traces of Functions

- Like we did with quadric surfaces, we can identify a function in several variables by looking at its **traces**, i.e., its intersections with planes of the form $x_i = C$ for some real number C.
- For instance, the vertical traces of the function $f(x, y) = x \sin y$ are the sine curves $f(C, y) = C \sin y$ of amplitude C in the plane $x = C$ and the lines $f(x, C) = x \sin C$ of slope $\sin C$ in the plane $y = C$.
- **Level curves** are obtain by taking the horizontal traces of a function of several variables and projecting them into the xy-plane.
- We have already seen that the level curves of the elliptic paraboloid $f(x, y) = x^2 + 3y^2$ are ellipses; the vertical traces are parabolas.

Level Curves and Traces of Functions

- We can collect all of the level curves of a function $f(x, y)$ in a plot in the xy-plane called a **contour map**.

Level Curves and Traces of Functions

- We can collect all of the level curves of a function $f(x, y)$ in a plot in the xy-plane called a **contour map**.
- For instance, the contour map of the elliptic paraboloid $x^2 + 3y^2$ consists of the ellipses $x^2 + 3y^2 = C$ for equally spaced C, e.g., $C = 0, 3, 6, 9$.

Level Curves and Traces of Functions

- We can collect all of the level curves of a function $f(x, y)$ in a plot in the xy-plane called a **contour map**.
- For instance, the contour map of the elliptic paraboloid $x^2 + 3y^2$ consists of the ellipses $x^2 + 3y^2 = C$ for equally spaced C, e.g., $C = 0, 3, 6, 9$. We refer to the constant C as the **contour interval**.

Level Curves and Traces of Functions

- We can collect all of the level curves of a function $f(x, y)$ in a plot in the xy-plane called a **contour map**.
- For instance, the contour map of the elliptic paraboloid $x^2 + 3y^2$ consists of the ellipses $x^2 + 3y^2 = C$ for equally spaced C, e.g., $C = 0, 3, 6, 9$. We refer to the constant C as the **contour interval**.
- **Caution:** Functions of more than two variables do not have level curves; rather, they have level surfaces.

Level Curves and Traces of Functions

- We can collect all of the level curves of a function $f(x, y)$ in a plot in the xy-plane called a **contour map**.
- For instance, the contour map of the elliptic paraboloid $x^2 + 3y^2$ consists of the ellipses $x^2 + 3y^2 = C$ for equally spaced C, e.g., $C = 0, 3, 6, 9$. We refer to the constant C as the **contour interval**.
- **Caution:** Functions of more than two variables do not have level curves; rather, they have level surfaces. One way to see this is that the points $(x, y, f(x, y))$ give rise to a three-dimensional object, hence the points $(x, y, z, f(x, y, z))$ give a four-dimensional object, and intersecting it with a plane gives a three-dimensional object.

Level Surfaces

Describe the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$.

(a.) elliptic parabolas

(b.) hyperboloids

(c.) hyperbolic parabolas

(d.) ellipsoids

Level Surfaces

Describe the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$.

(a.) elliptic parabolas

(b.) hyperboloids

(c.) hyperbolic parabolas

(d.) ellipsoids

We note that $x^2 + y^2 + z^2 = C$ gives a sphere of radius $\sqrt{C}$ for each nonnegative real number C. Given a real number $C < 0$, the level curves vanish since $x^2 + y^2 + z^2 \geq 0$ for all points (x, y, z) in $\mathbb{R}^3$.