Arc Length

- Given a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ that is continuously differentiable on $[a, b]$, we have that the length of the path traced by $\mathbf{r}(t)$ on $[a, b]$ is given by

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- Generally, we can form the **arc length** function of $r(t)$ for $t \geq a$ by

$$s(t) = \int_a^t ||\mathbf{r}'(u)||\, du = \int_a^t \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2}\, du.$$

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$$\text{speed } = \frac{d}{dt}s(t) = \frac{d}{dt}\int_a^t ||\mathbf{r}'(u)||\, du = ||\mathbf{r}'(t)||$$

by the Fundamental Theorem of Calculus.

Arc Length

True (a.) or False (b.)

The arc length of $\mathbf{r}(t) = \langle \cos 3t, \sin 3t, 3t \rangle$ on $[0, 2\pi]$ is $2\pi\sqrt{18}$.

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- We say that an **arc length parametrization** is one for which $s'(t) = ||\mathbf{r}'(t)|| = 1$ for all $t \geq a$. We think of an arc length parametrization of a curve as one for which we can walk from point a to point b at a constant speed of 1 unit per second.

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  3. Our arc length parametrization will be $\bar{\mathbf{r}}(s) = \mathbf{r}(g^{-1}(s))$.
- We refer to $t = g^{-1}(s)$ as the **arc length parameter**.
- Further, we say that $\mathbf{r}(t)$ is **parametrized by arc length** for all $t \geq a$ whenever we have that $||\mathbf{r}'(t)|| = 1$ for all $t \geq a$.

Computing the Arc Length of a Curve

Choose the arc length parametrization of the curve $\mathbf{r}(t) = \langle \cos 4t, \sin 4t, 3t \rangle$ for all $t \geq 0$.

(a.) $\bar{\mathbf{r}}(s) = \left\langle \frac{4s}{\sqrt{41}}, \frac{4s}{\sqrt{41}}, \frac{3s}{\sqrt{41}} \right\rangle$

(b.) $\bar{\mathbf{r}}(s) = \left\langle \frac{\cos(4s)}{2}, \frac{\sin(4s)}{2}, \frac{s\sqrt{2}}{2} \right\rangle$

(c.) $\bar{\mathbf{r}}(s) = \left\langle \frac{\cos s}{2}, \frac{\sin s}{2}, \frac{s\sqrt{2}}{2} \right\rangle$

(d.) $\bar{\mathbf{r}}(s) = \left\langle \cos\left(\frac{4s}{5}\right), \sin\left(\frac{4s}{5}\right), \frac{3s}{5} \right\rangle$