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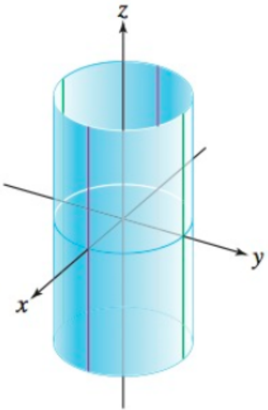
- Considering these conic sections as three-dimensional objects by letting z vary, we obtain the family of **cylinders**.

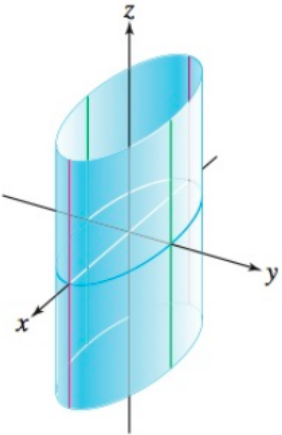
Cylinders



$$x^2 + y^2 = r^2$$

Right-circular cylinder of radius r

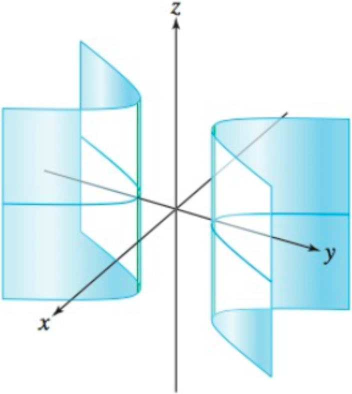
Cylinders



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Elliptic cylinder

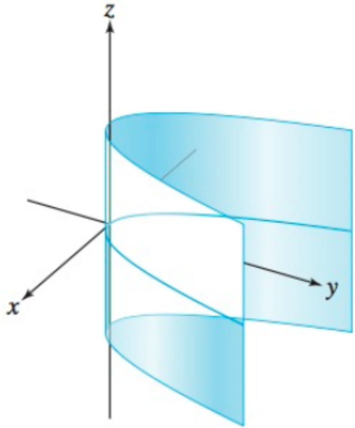
Cylinders



$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

Hyperbolic cylinder

Cylinders



$y = ax^2$

Parabolic cylinder

Basics of Quadric Surfaces

- **Quadric surfaces** are three-dimensional generalizations of the conic sections of the (two-dimensional) Cartesian plane.

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- Given that $D = E = F = 0$, we say that a quadric surface is in **standard position** or of **standard form**.
- Quadric surfaces are uniquely determined by their **traces**, i.e., the conic sections that are obtained by intersecting a quadric surface with a plane that is parallel to one of the three coordinate planes.

Common Examples of Quadric Surfaces

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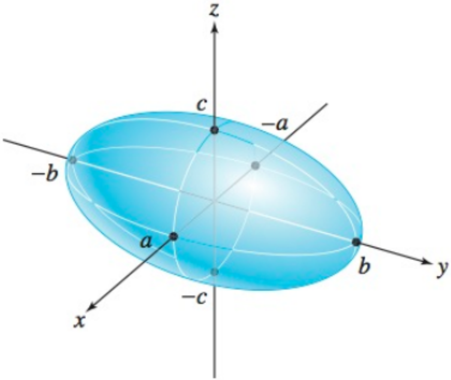
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

  Given that $a = b = c = r$, we obtain the ellipsoid $x^2 + y^2 + z^2 = r^2$, i.e., a sphere centered at the origin $(0, 0, 0)$ with radius $r > 0$.

Common Examples of Quadric Surfaces



DF **FIGURE 1** Ellipsoid with equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

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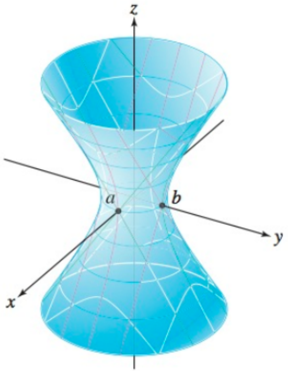
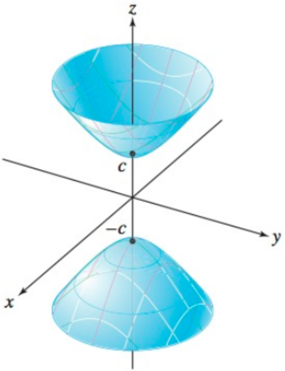
Common Examples of Quadric Surfaces



(A) Hyperboloid of one sheet



(B) Hyperboloid of two sheets

Common Examples of Quadric Surfaces

- **Elliptic cones** are quadric surfaces whose xy-traces are ellipses and whose yz- and xz-traces are pairs of diagonal lines.

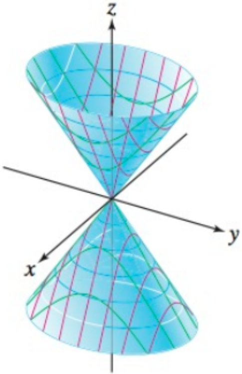
Common Examples of Quadric Surfaces

- **Elliptic cones** are quadric surfaces whose *xy*-traces are ellipses and whose *yz*- and *xz*-traces are pairs of diagonal lines.

- We can describe an elliptic cone in standard position by an equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2.$$

Common Examples of Quadric Surfaces



DF **FIGURE 7** Elliptic cone

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2.$$

Common Examples of Quadric Surfaces

- **Elliptic paraboloids** are quadric surfaces whose *xy*-traces are ellipses and whose *yz*- and *xz*-traces are upward-opening parabolas.

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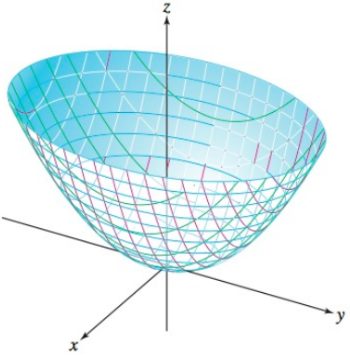
- We can describe an elliptic paraboloid in standard position by

$$z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2.$$

Common Examples of Quadric Surfaces



(A) Elliptic paraboloid

$$z = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2$$

FIGURE 8

Common Examples of Quadric Surfaces

- **Hyperbolic paraboloids** are quadric surfaces whose xy-traces are hyperbolas, whose yz-traces are downward-opening parabolas, and whose xz-traces are upward-opening parabolas.

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- **Hyperbolic paraboloids** are quadric surfaces whose *xy*-traces are hyperbolas, whose *yz*-traces are downward-opening parabolas, and whose *xz*-traces are upward-opening parabolas.
- We can describe a hyperbolic paraboloid in standard position by

$$z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2.$$

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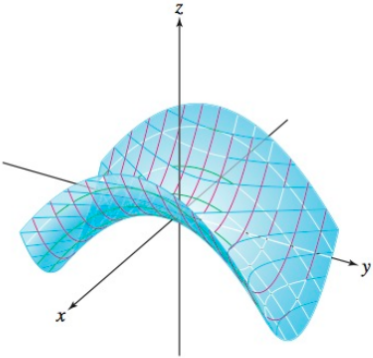

(B) Hyperbolic paraboloid

$$z = \left(\frac{x}{2}\right)^2 - \left(\frac{y}{3}\right)^2$$

Common Examples of Quadric Surfaces

True (a.) or False (b.)

Every trace of an ellipsoid is an ellipse.

True (a.) or False (b.)

Every trace of an ellipsoid is an ellipse.

(a.) True.

Common Examples of Quadric Surfaces

True (a.) or False (b.)

Every trace of a hyperboloid is a hyperbola.

Common Examples of Quadric Surfaces

True (a.) or False (b.)

Every trace of a hyperboloid is a hyperbola.

(b.) False. Both vertical traces of a hyperboloid are hyperbolas, but the horizontal trace (xy-trace) of a hyperboloid is an ellipse.

True (a.) or False (b.)

There exists a quadric surface whose horizontal and vertical traces are each parabolas.

Common Examples of Quadric Surfaces

True (a.) or False (b.)

There exists a quadric surface whose horizontal and vertical traces are each parabolas.

(b.) False.

True (a.) or False (b.)

There exists a quadric surface whose horizontal and vertical traces are each parabolas.

(b.) False. On the contrary, if it were possible, then we would have xy-trace $y = a_0 x^2$, yz-trace $z = b_0 y^2$, and xz-trace $x = c_0 z^2$, hence

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + ax + by + cz + d = 0$$

is not a quadratic equation — a contradiction.

Common Examples of Quadric Surfaces

Finding the Form of a Quadric Surface

Classify the quadric surface in standard position with horizontal trace given by the equation $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$ and vertical traces given by the equations $\left(\frac{y}{4}\right)^2 + \left(\frac{z}{6}\right)^2 = 1$ and $\left(\frac{x}{2}\right)^2 + \left(\frac{z}{6}\right)^2 = 1$.

(a.) right-circular cylinder

(b.) elliptic cylinder

(c.) ellipsoid

(d.) elliptic paraboloid

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(a.) right-circular cylinder

(b.) elliptic cylinder

(c.) ellipsoid

(d.) elliptic paraboloid

Each of the traces is an ellipse, hence the quadric surface is an ellipsoid.

Finding the Form of a Quadric Surface

Classify the quadric surface in standard position with horizontal trace given by the equation $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$ and vertical traces given by the equations $\left(\frac{y}{6}\right)^2 - \left(\frac{z}{\sqrt{27}}\right)^2 = 1$ and $\left(\frac{x}{4}\right)^2 - \left(\frac{z}{\sqrt{27}}\right)^2 = 1$.

(a.) hyperbolic cylinder

(b.) hyperbolic paraboloid

(c.) hyperboloid of two sheets

(d.) hyperboloid of one sheet

Common Examples of Quadric Surfaces

Finding the Form of a Quadric Surface

Classify the quadric surface in standard position with horizontal trace given by the equation $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$ and vertical traces given by the equations $\left(\frac{y}{6}\right)^2 - \left(\frac{z}{\sqrt{27}}\right)^2 = 1$ and $\left(\frac{x}{4}\right)^2 - \left(\frac{z}{\sqrt{27}}\right)^2 = 1$.

(a.) hyperbolic cylinder

(b.) hyperbolic paraboloid

(c.) hyperboloid of two sheets

(d.) hyperboloid of one sheet

Each of the vertical traces is a hyperbola, and the horizontal trace is an ellipse, hence this is a hyperboloid; moreover, there exist points corresponding to $z = 0$, from which it follows that there is one sheet.