MATH 126: Laboratory Workbook Solutions

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Review of Limits and Continuity

Example 1. Compute the limit of $f(x) = x^2$ as x approaches $a = 1$.

Solution. Computing the limit is essentially like playing a game of limbo. We are first handed a real number $\varepsilon > 0$. Our challenge is then to find a real number $\delta > 0$ such that $|x^2 - 1| < \varepsilon$ whenever we assume that $|x - 1| < \delta$. Of course, we are at liberty to take δ as small as necessary to ensure that $|x^2 - 1| < \varepsilon$. We may therefore assume that $0 < \delta \leq 1$. Considering that $x^2 - 1 = (x - 1)(x + 1)$, if we assume that $|x - 1| < \delta \leq 1$, then we must have that $0 < x < 2$, from which it follows that $|x + 1| \leq |x| + 1 = x + 1 < 3$ by the Triangle Inequality. Consequently, we have that

$$|x^2 - 1| = |(x - 1)(x + 1)| = |x - 1||x + 1| < 3\delta,$$

and if we wish to have that $|x^2 - 1| < \varepsilon$, then we should choose $\delta = \min\{1, \varepsilon/3\}$. $\diamond$

Example 1, Revisited. Compute the limit of $f(x) = x^2$ as x approaches $a = 1$.

Solution. Using the graph of $f(x) = x^2$, we find that the limit is 1. Particularly, if we trace the graph with our left pointer finger moving from left to right toward the point $x = 1$, our finger stops at $y = 1$. Likewise, if we trace the graph with our right pointer finger moving from right to left toward $x = 1$, our finger stops at $y = 1$. Put in calculus language, we have that $L^- = 1 = L^+$. $\diamond$

Example 2. Prove that the function $f(x) = |x|$ is continuous for all real numbers a.

Proof. Using the definition of $|x|$, we have that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \text{ and} \\ -x & \text{if } x < 0. \end{cases}$$

Consequently, it suffices to show that $g(x) = x$ and $h(x) = -x$ are continuous. Given real numbers $\varepsilon_1, \varepsilon_2 > 0$, our challenge is to find real numbers $\delta_1, \delta_2 > 0$ such that $|x - a| < \varepsilon_1$ whenever $|x - a| < \delta_1$ and $|-x - (-a)| < \varepsilon_2$ whenever $|x - a| < \delta_2$. Our best bet is to choose $\delta_1 = \varepsilon_1$ and $\delta_2 = \varepsilon_2$.

We have shown that $g(x) = x$ and $h(x) = -x$ are continuous for all real numbers a, so $|x|$ is continuous for all nonzero real numbers. We are done as soon as we show that

$$\lim_{x \to 0^-} |x| = 0 = \lim_{x \to 0^+} |x|.$$

By continuity of the functions $g(x)$ and $h(x)$ and by definition of $|x|$, the left-hand limit is given by $\lim\limits_{x \to 0^-} h(x) = h(0) = 0$, and the right-hand limit is given by $\lim\limits_{x \to 0^+} g(x) = g(0) = 0$. $\square$

Example 2, Revisited. Prove that the function $f(x) = |x|$ is continuous for all real numbers a.

Proof. Observe that we can graph $|x|$ without lifting our pencil, hence $|x|$ is continuous. $\square$

Review of Derivatives and L'Hôpital's Rule

Example 3. Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = x^2$.

Solution. Our first order of business is to simplify the difference quotient $D_x(h)$. We have that

$$D_x(h) = \frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h}.$$

By definition of the limit as $h \to 0$, it follows that $h \neq 0$, hence we conclude that

$$f'(x) \overset{\text{def}}{=} \lim_{h \to 0} D_x(h) = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{2x + h}{1} = 2x,$$

where the third equals sign follows by cancelling a factor of h in the numerator and denominator. $\diamond$

Example 4. Compute the limit of $f(x) = \dfrac{\ln x}{x^3 - 1}$ as x approaches $a = 1$.

Solution. Considering that $\ln 1 = 0$ and $1^3 - 1 = 0$, it follows that

$$\lim_{x \to 1} f(x) = \frac{0}{0}.$$

By L'Hôpital's Rule, therefore, we have that

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{\ln x}{x^3 - 1} \overset{\text{L'H}}{=} \lim_{x \to 1} \frac{1/x}{3x^2} = \lim_{x \to 1} \frac{1}{3x^3} = \frac{1}{3}.$$ $\diamond$

Example 5. Given that $\dfrac{d}{dx} \sin x = \cos x$, compute the limit of $f(x) = \dfrac{\sin x}{x}$ as x approaches $a = 0$.

Solution. Considering that $\sin 0 = 0$, it follows that

$$\lim_{x \to 0} f(x) = \frac{0}{0}.$$

By L'Hôpital's Rule, therefore, we have that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} \overset{\text{L'H}}{=} \lim_{x \to 0} \frac{\cos x}{1} = 1.$$ $\diamond$

Caution: Unfortunately, this is not a valid proof of this limit identity. In fact, this limit identity is needed to prove that $\frac{d}{dx} \sin x = \cos x$. In order to prove this identity in a rigorous and non-circular manner, we must use tools from trigonometry and the Squeeze Theorem.

Example 6. Compute the limit of $f(x) = (2x - \pi) \sec x$ as x approaches $a = \dfrac{\pi}{2}$ from the left.

Solution. Considering that $\lim\limits_{x \to \frac{\pi}{2}^-} \cos x = 0$ and $\lim\limits_{x \to \frac{\pi}{2}^-} (2x - \pi) = 0$, it follows that

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} (2x - \pi) \sec x = \lim_{x \to \frac{\pi}{2}^-} \frac{2x - \pi}{\cos x} = \frac{0}{0}.$$

By L'Hôpital's Rule, therefore, we have that

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} (2x - \pi) \sec x = \lim_{x \to \frac{\pi}{2}^-} \frac{2x - \pi}{\cos x} \overset{\text{L'H}}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{2}{-\sin x} = \frac{2}{-1} = -2.$$ $\diamond$

Example 7. Compute the limit of $f(x) = \dfrac{\sin x}{\sin x + \tan x}$ as x approaches $a = 0$.

Solution. Considering that $\sin(0) = 0$ and $\sin(0) + \tan(0) = 0$, it follows that

$$\lim_{x \to 0} f(x) = \frac{0}{0}.$$

By L'Hôpital's Rule, therefore, we have that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{\sin x + \tan x} \overset{\text{L'H}}{=} \lim_{x \to 0} \frac{\cos x}{\cos x + \sec^2 x} = \frac{1}{1+1} = \frac{1}{2};$$

however, it is not necessary to use L'Hôpital's Rule. By definition of the limit, we have that $x \neq 0$, hence we may cancel each factor of $\sin x$ from the numerator and denominator of $f(x)$ to obtain

$$f(x) = \frac{\sin x}{\sin x + \tan x} = \frac{\sin x}{\sin x(1 + \sec x)} = \frac{1}{1 + \sec x}.$$

Of course, we have that $\sec 0 = 1$, hence the limit can be evaluated directly after simplification. $\diamond$

Integration and Improper Integrals

Example 8. Prove that the function $F(x) = \dfrac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$.

Proof. Observe that we have $F'(x) = \frac{1}{3} \cdot 3x^2 = x^2 = f(x)$ by the Power Rule. $\square$

Example 9. Compute the antiderivative of $f(x) = \dfrac{1}{x}$.

Solution. We note that $\frac{d}{dx} \ln x = \frac{1}{x}$ whenever $x > 0$ and $\frac{d}{dx} \ln(-x) = \frac{1}{x}$ whenever $x < 0$, hence we have that $F(x) = \ln|x|$ is an antiderivative of $f(x)$. We conclude that $\int \frac{1}{x}\, dx = \ln|x| + C$. $\diamond$

Example 10. Compute the antiderivative of $f(x) = \sin x \cos x$.

Solution. By the Chain Rule for integration (AKA the Substitution Rule), we find that

$$\int \sin x \cos x \, dx = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\sin x)^2 + C,$$

where we use $u = \sin x$ and the fact that $\frac{du}{dx} = \cos x$ so that $du = \cos x \, dx$. $\diamond$

Example 11. Compute the antiderivative of $f(x) = xe^{x^2}$.

Solution. By the Chain Rule for integration (AKA the Substitution Rule), we find that

$$\int xe^{x^2} \, dx = \int \frac{1}{2}e^u \, du = \frac{1}{2} \int e^u \, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C,$$

where we use $u = x^2$ so that $du = 2x \, dx$ and $\frac{1}{2}du = x \, dx$. $\diamond$

Example 12. Compute the (signed) area under the curve $f(x) = x^3$ from $x = 0$ to $x = 1$.

Solution. One interpretation of the definite integral $\int_0^1 x^3 \, dx$ is as the (signed) area between x^3 and the x-axis from $x = 0$ to $x = 1$. Considering that x^3 is above the x-axis from $x = 0$ to $x = 1$, this is the (signed) area under the curve from $x = 0$ to $x = 1$. Certainly, we have that $F(x) = \frac{1}{4}x^4$ is an antiderivative of $f(x) = x^3$ by the Power Rule, hence by the FToC I, we conclude that

$$\int_0^1 x^3 \, dx = \frac{1}{4}x^4 \bigg|_0^1 = \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4 = \frac{1}{4}. \qquad \diamond$$

Example 13. Compute the (signed) area between $f(x) = \sin x$ and the x-axis on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Solution. Observe that $F(x) = -\cos x$ is an antiderivative of $f(x) = \sin x$, hence by the FToC I,

$$\int_{-\pi/2}^{\pi/2} \sin x \, dx = -\cos x \bigg|_{-\pi/2}^{\pi/2} = -\cos(\pi/2) + \cos(-\pi/2) = -0 + 0 = 0. \qquad \diamond$$

Example 14. Given a differentiable function $g(x)$, use the Fundamental Theorem of Calculus and the Chain Rule for derivatives to prove that

$$\frac{d}{dx} \int_a^{g(x)} f(t) \, dt = f'(g(x))g'(x).$$

Proof. By the FToC II, we have that $I(x) = \int_a^x f(t) \, dt$ is a differentiable function with derivative $I'(x) = f(x)$. Considering that $g(x)$ is differentiable by hypothesis, we have that $I(g(x))$ is a differentiable function. Once again, by the FToC II and the Chain Rule, we conclude that

$$\frac{d}{dx} \int_a^{g(x)} f(t) \, dt = \frac{d}{dx} I(g(x)) = I'(g(x)) \cdot g'(x) = f'(g(x)) \cdot g'(x). \qquad \square$$

Example 15. Compute the improper integral $\int_1^\infty x^{-2} \, dx$.

Solution. By the Power Rule, we have that $F(x) = -x^{-1}$ is an antiderivative of $f(x) = x^{-2}$ so that

$$\int_1^\infty x^{-2} \, dx = \lim_{b \to \infty} \int_1^b x^{-2} = \lim_{b \to \infty} [-b^{-1} + (1)^{-1}] = \lim_{b \to \infty} \left(1 - \frac{1}{b}\right) = 1. \qquad \diamond$$

Example 16. Compute the improper integral $\int_{-\infty}^1 e^x \, dx$.

Solution. Considering that $F(x) = e^x$ is an antiderivative of $f(x) = e^x$, we have that

$$\int_{-\infty}^1 e^x \, dx = \lim_{a \to -\infty} \int_a^1 e^x \, dx = \lim_{a \to -\infty} [e^1 - e^a] = e^1 = e. \qquad \diamond$$

Example 17. Compute the improper integral $\int_{-\infty}^\infty xe^{-x^2} \, dx$.

Solution. By Example 11, we have that $F(x) = -\frac{1}{2}e^{-x^2}$ is an antiderivative of $f(x) = xe^{-x^2}$ so that

$$\int_{-\infty}^\infty xe^{-x^2} \, dx = \lim_{b \to \infty} \lim_{a \to -\infty} \int_a^b xe^{-x^2} \, dx = \lim_{b \to \infty} \lim_{a \to -\infty} \left[-\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-a^2}\right] = \lim_{b \to \infty} -\frac{1}{2}e^{-b^2} = 0. \qquad \diamond$$

Example 18. Compute the improper integral $\int_0^1 (x-1)^{-1}\,dx$.

Solution. Considering that $\lim_{x\to 1^-}(x-1)^{-1} = -\infty$, we are dealing with an improper integral with a vertical asymptote. Using the substitution $u = x-1$, we have that $du = dx$, $u|_{x=1} = 0$, and $u|_{x=0} = -1$. Consequently, we have that $\lim_{u\to 0^-} u^{-1} = -\infty$. Observe that $F(u) = \ln(-u)$ is an antiderivative of $f(u) = u^{-1}$ whenever $u < 0$, hence we may evaluate the improper integral by

$$\int_0^1 (x-1)^{-1}\,dx = \int_{-1}^0 u^{-1}\,du = \lim_{b\to 0^-}\int_{-1}^b u^{-1}\,du = \lim_{b\to 0^-}[\ln(-b) - \ln(1)] = -\infty.$$

Ultimately, we find that the improper integral does not converge. ◇

Example 19. Compute the improper integral $\int_0^1 x^{-1/2}\,dx$.

Solution. Considering that $\lim_{x\to 0^+} x^{-1/2} = \infty$, we are dealing with an improper integral with a vertical asymptote. Observe that $F(x) = 2x^{1/2}$ is an antiderivative of $f(x) = x^{-1/2}$ whenever we have that $x > 0$, hence we may evaluate the improper integral by

$$\int_0^1 x^{-1/2}\,dx = \lim_{a\to 0^+}\int_a^1 x^{-1/2}\,dx = \lim_{a\to 0^+}[2(1)^{1/2} - 2(a)^{1/2}] = 2.$$ ◇

Example 20. Compute the improper integral $\int_{-1}^1 x^{-2/3}\,dx$.

Solution. Considering that $\lim_{x\to 0^-} x^{-2/3} = -\infty$ and $\lim_{x\to 0^+} x^{-2/3} = \infty$, we are dealing with an improper integral with a vertical asymptote. Observe that $F(x) = 3x^{1/3}$ is an antiderivative of $f(x) = x^{-2/3}$ whenever $x \neq 0$, hence we may evaluate the improper integral by

$$\begin{aligned}\int_{-1}^1 x^{-2/3}\,dx &= \int_{-1}^0 x^{-2/3}\,dx + \int_0^1 x^{-2/3}\,dx \\ &= \lim_{b\to 0^-}\int_{-1}^b x^{-2/3}\,dx + \lim_{a\to 0^+}\int_a^1 x^{-2/3}\,dx \\ &= \lim_{b\to 0^-}[3(b)^{1/3} - 3(-1)^{1/3}] + \lim_{a\to 0^+}[3(1)^{1/3} - 3(a)^{1/3}] = 3 + 3 = 6.\end{aligned}$$ ◇

Example 21. Determine the convergence or divergence of the improper integral $\int_0^\infty xe^x\,dx$.

Solution. We do not yet have the tools to evaluate this improper integral directly, hence we use the Comparison Theorem. Observe that $xe^x \geq e^x \geq 0$ for each real number $x \geq 0$. Considering that

$$\int_0^\infty e^x\,dx = \lim_{b\to\infty}\int_0^b e^x\,dx = \lim_{b\to\infty}[e^b - 1] = \infty$$

so that $\int_0^\infty e^x\,dx$ diverges, it follows by the Comparison Theorem that $\int_0^\infty xe^x\,dx$ diverges. ◇

Example 22. Determine the convergence or divergence of the improper integral $\int_0^\infty x^{-2}\sin^2 x\,dx$.

Solution. Unfortunately, there is not an elementary antiderivative for this function, so there is no hope for us to find an exact value; however, we have that $x^{-2}\sin^2 x$ is continuous on $(0,1]$, hence it is enough to show that $x^{-2}\sin^2 x$ is bounded by another function on $[1,\infty)$. Observe that $0 \leq \sin^2 x \leq 1$ for all x so that $0 \leq x^{-2}\sin^2 x \leq x^{-2}$ for all $x \geq 1$. By Example 15, we have that $\int_1^\infty x^{-2}\,dx = 1$, hence we conclude by the Comparison Theorem that $\int_0^\infty x^{-2}\sin^2 x\,dx$ converges. ◇

Sequences

Example 23. Find an explicit formula $a_n = f(n)$ for the infinite sequence $-1, 1, -1, 1, \dots$ that alternates between -1 and 1. Be sure to specify the index set N, e.g., $N = \{n \in \mathbb{N} \mid n \geq 1\}$.

Solution. Observe that $(-1)^1 = -1$, $(-1)^2 = 1$, $(-1)^3 = -1$, and $(-1)^4 = 1$, hence we suspect that $f(n) = (-1)^n$ for all integers $n \geq 1$. Given any positive integer k, we have that

$$(-1)^{2k} = [(-1)^2]^k = 1^k = 1 \text{ and } (-1)^{2k+1} = (-1)(-1)^{2k} = (-1)(1) = -1.$$

Our proof is complete because every integer is equal $2k$ or $2k+1$ for some integer k. $\diamond$

Example 24. Find an explicit formula $a_n = f(n)$ for the infinite sequence that starts $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, etc. Be sure to specify the index set N, e.g., $N = \{n \in \mathbb{N} \mid n \geq 1\}$.

Solution. Observe that $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, and $2^4 = 16$, hence we conclude that $f(n) = \frac{1}{2^{n-1}}$ for all integers $n \geq 1$. We could also write $f(n) = \frac{1}{2^n}$ for all integers $n \geq 0$. $\diamond$

Example 25. Find a closed form for the sequence $a_n = 2a_{n-1}$ for each integer $n \geq 2$ and $a_1 = 1$.

Solution. We have that $a_n = 2a_{n-1} = 2 \cdot 2a_{n-2} = \cdots = \underbrace{2 \cdot 2 \cdots 2}_{n-1 \text{ factors}} a_1 = 2^{n-1}$ for each integer $n \geq 2$. $\diamond$

Example 26. Compute the limit of the sequence $a_n = \dfrac{1}{2^n}$, or prove that it does not exist.

Solution. Considering that a_n is a subsequence of $b_n = \frac{1}{n}$, we suspect that $\lim\limits_{n \to \infty} a_n = 0 = \lim\limits_{n \to \infty} b_n$. Given any real number $\varepsilon > 0$, if we have that $n > M = \log_2\left(\frac{1}{\varepsilon}\right)$, then

$$\left|\frac{1}{2^n}\right| = \frac{1}{2^n} < \frac{1}{2^M} = \frac{1}{2^{\log_2(1/\varepsilon)}} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon. \qquad \diamond$$

Example 27. Compute the limit of the sequence $a_n = (-1)^n$, or prove that it does not exist.

Solution. Generally, the best way to determine that a sequence diverges is to find two subsequences that converge to different limits. Using this technique, the constant subsequences $a_{2k} = (-1)^{2k} = 1$ and $a_{2k+1} = (-1)^{2k+1} = -1$ converge to 1 and -1, respectively, hence a_n is divergent. $\diamond$

Certainly, we can also attack this problem with tools that we have already established.

Solution. By definition of the limit, if it were true that $\lim\limits_{n \to \infty} a_n = L$, then there would exist a positive real number M such that for all $n > M$, we would have that $|(-1)^n - L| < 1$ so that $L - 1 < (-1)^n < L + 1$. But then, we would have that $L - 1 < (-1)^{n+1} = -(-1)^n < L + 1$ so that $-(L+1) < (-1)^n < -(L-1) < L - 1$. Put another way, we have that $(-1)^n > L - 1$ and $(-1)^n < L - 1$. Clearly, this is impossible. We conclude therefore that a_n diverges. $\diamond$

Example 28. Compute the limit of the sequence $a_n = \dfrac{\sin n}{n}$.

Solution. By the previous fact, we have that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin n}{n} = \lim_{x\to\infty} \frac{\sin x}{x} = 0$$

by the Squeeze Theorem. Particularly, we have that $-1 \leq \sin x \leq 1$ so that $-\dfrac{1}{x} \leq \dfrac{\sin x}{x} \leq \dfrac{1}{x}$. ◇

Example 29. Compute the limit of the sequence $a_n = \dfrac{\ln n}{n}$.

Solution. By the previous fact, we have that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \stackrel{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$ ◇

Example 30. Compute the limit of the sequence $a_n = \dfrac{n^4 - 5n^3 + 3n^2 + 1}{3n^4 - 7n^2 + n + 1}$.

Solution. By the previous fact, we have that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^4 - 5n^3 + 3n^2 + 1}{3n^4 - 7n^2 + n + 1} = \lim_{x\to\infty} \frac{x^4 - 5x^3 + 3x^2 + 1}{3x^4 - 7x^2 + x + 1} = \lim_{x\to\infty} \frac{1 - \frac{5}{x} + \frac{3}{x^2} + \frac{1}{x^4}}{3 - \frac{7}{x^2} + \frac{1}{x^3} + \frac{1}{x^4}} = \frac{1}{3}.$$ ◇

Example 31. Determine if the sequence $a_n = \dfrac{1}{3^{3n-2}}$ is geometric. If so, find the constant c and the common ratio r, and determine with justification if a_n converges or diverges; if not, explain why.

Solution. Observe that we can write $3^{3n-2} = 3^{3n} \cdot 3^{-2} = \frac{1}{3^2} \cdot (3^3)^n = \frac{1}{9} \cdot 27^n$. Consequently, we have that $a_n = 9 \cdot \frac{1}{27^n} = 9 \cdot \left(\frac{1}{27}\right)^n$ is a geometric sequence with $c = 9$ and $r = \frac{1}{27}$. Considering that $0 \leq r < 1$, it follows that a_n converges. ◇

Example 32. Determine if the sequence $6, -3, \frac{3}{2}, -\frac{3}{4}$, etc. is geometric. If so, find the constant c and the common ratio r, and determine with justification its convergence; if not, explain why.

Solution. Observe that we can write $a_1 = 6 = \frac{-12}{-2}$, $a_2 = -3 = \frac{-12}{(-2)^2}$, $a_3 = \frac{3}{2} = \frac{-12}{(-2)^3}$, and $a_4 = -\frac{3}{4} = \frac{-12}{(-2)^4}$. Consequently, we have that $a_n = -12 \cdot \frac{1}{(-2)^n} = -12 \cdot \left(-\frac{1}{2}\right)^n$ is a geometric sequence with $c = -12$ and $r = -\frac{1}{2}$. Considering that $-1 < r < 0$, it follows that a_n converges. ◇

Example 33. Determine if the sequence $a_n = \ln e^\pi$ is geometric. If so, find the constant c and the common ratio r, and determine with justification if a_n converges or diverges; if not, explain why.

Solution. Observe that we can write $a_n = \ln e^\pi = \pi \ln e = \pi = \pi \cdot 1^n$. Consequently, we have that a_n is a geometric sequence with $c = \pi$ and $r = 1$ so that a_n converges. ◇

Example 34. Prove that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof. Certainly, we have that $-|a_n| \leq a_n \leq |a_n|$ for all integers $n \geq 1$. By hypothesis, we have that $\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = 0$. We conclude by the Squeeze Theorem that $\lim_{n\to\infty} a_n = 0$. □

Example 35. Given any real number $c \neq 0$, prove that the geometric sequence $a_n = cr^n$ satisfies

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} cr^n = \begin{cases} 0 & \text{if } r < 0 \text{ and } |r| < 1 \text{ and} \\ \text{DNE} & \text{if } r \leq -1. \end{cases}$$

Proof. Given that $r < 0$ and $|r| < 1$, it follows that $\lim_{n\to\infty} |r|^n = 0$. By the previous fact, therefore, we conclude that $\lim_{n\to\infty} cr^n = 0$. Given that $r = -1$, it follows that $cr^n = c(-1)^n$ oscillates between $-c$ and c, hence cr^n diverges. Given that $r < -1$, it follows that $|r| = 1 + \varepsilon$ for some positive real number ε. Considering that $(1+\varepsilon)^n$ grows arbitrarily large, we have that $|c||r|^n = |cr^n|$ grows arbitrarily large so that cr^n is unbounded and therefore diverges. $\square$

Example 36. Given any real number c, prove that $\lim_{n\to\infty} \dfrac{c^n}{n!} = 0$.

Proof. Of course, if $c = 0$, then this is clear because the sequence is constantly 0. Consider the case that $M \leq c < M + 1$ for some integer $M \geq 0$. Consider the nth term of the sequence.

$$\frac{c^n}{n!} = \frac{c \cdot c \cdot c \cdots c \cdot c \cdot c \cdots c}{1 \cdot 2 \cdots M \cdot (M+1) \cdot (M+2) \cdots (n-1) \cdot n} = \underbrace{\frac{c}{1} \cdot \frac{c}{2} \cdot \frac{c}{3} \cdots \frac{c}{M}}_{\text{Call this constant } R.} \cdot \underbrace{\frac{c}{M+1} \cdot \frac{c}{M+2} \cdots \frac{c}{n-1}}_{\text{Each term here is } \leq 1.} \cdot \frac{c}{n}.$$

Consequently, we have that $0 \leq \dfrac{c^n}{n!} \leq R \cdot \dfrac{c}{n}$. Considering that $\lim_{n\to\infty} R \cdot \dfrac{c}{n} = R \cdot c \cdot \lim_{n\to\infty} \dfrac{1}{n} = R \cdot c \cdot 0 = 0$, we conclude that $\lim_{n\to\infty} \dfrac{c^n}{n!} = 0$ by the Squeeze Theorem.

Given that $M \leq c < M+1$ for some integer $M \leq -1$, we have that $|M+1| < |c| \leq |M|$, and we can apply a similar strategy as in the previous paragraph with these positive integers to find that $0 \leq \left|\dfrac{c^n}{n!}\right| = \dfrac{|c|^n}{n!} \leq R \cdot \dfrac{|c|}{n}$. By the Squeeze Theorem and Exercise 34, our proof is complete. $\square$

Example 37. Compute the limit of the sequence $a_n = \sin(e^{-n})$.

Solution. Observe that $a_n = f(b_n)$ for the continuous function $f(x) = \sin x$ and geometric sequence $b_n = e^{-n} = \left(\frac{1}{e}\right)^n$ with $\lim_{n\to\infty} b_n = 0$. We conclude that $\lim_{n\to\infty} a_n = \sin(\lim_{n\to\infty} b_n) = \sin(0) = 0$. $\diamond$

Example 38. Determine whether the sequence $a_n = \sin\left(\dfrac{1}{n}\right)$ is monotone.

Solution. Observe that $a_n = f(n)$ for the differentiable function $f(x) = \sin\left(\dfrac{1}{x}\right)$. Considering that

$$f'(x) = \cos\left(\frac{1}{x}\right) \cdot -\frac{1}{x^2} < 0 \text{ for all } x > \frac{2}{\pi},$$

we conclude that a_n is decreasing for $n \geq 1$, hence a_n is monotone for all $n \geq 1$. $\diamond$

Example 39. Determine whether the sequence $a_n = -ne^{-n^2}$ is monotone.

Solution. Observe that $a_n = f(n)$ for the differentiable function $f(x) = -xe^{-x^2}$. Considering that

$$f'(x) = -e^{-x^2} - xe^{-x^2} \cdot -2x = -e^{-x^2} + 2x^2e^{-x^2} = e^{-x^2}(2x^2 - 1) > 0 \text{ for all } x > \frac{\sqrt{2}}{2},$$

we conclude that a_n is increasing for $n \geq 1$, hence a_n is monotone for all $n \geq 1$. $\diamond$

Example 40. Determine whether the sequence $a_n = \cos(\pi n)$ is monotone.

Solution. Observe that $a_n = f(n)$ for the differentiable function $f(x) = \cos(\pi x)$. Considering that $f'(x) = -\pi \sin x$ takes positive and negative values for infinitely many x, a_n is not monotone. $\diamond$

Example 41. Determine whether the sequence $a_n = \sin\left(\frac{1}{n}\right)$ converges. If so, find the limit.

Solution. Considering that $\sin x$ is a continuous function, we can easily evaluate the limit:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sin(0) = 0. \qquad \diamond$$

We could also have employed the Monotone Convergence Theorem to guarantee convergence.

Solution. Observe that $-1 \leq a_n \leq 1$, hence a_n is bounded. By Example 38, we have that a_n is monotone (decreasing) for all $n \geq 1$, hence a_n converges by the Monotone Convergence Theorem. $\diamond$

Example 42. Determine whether the sequence $a_n = -ne^{-n^2}$ converges. If so, find the limit.

Solution. Considering that $a_n = f(n)$ for the function $f(x) = -xe^{-x^2}$, we have that

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = -\lim_{x\to\infty} \frac{x}{e^{x^2}} \overset{\text{L'H}}{=} -\lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0. \qquad \diamond$$

We could also have employed the Monotone Convergence Theorem to guarantee convergence.

Solution. By Example 39, we have that a_n is monotone (increasing) for all $n \geq 1$. Consequently, we have that $a_1 = -e^{-1}$ is a lower bound for a_n. On the other hand, we have that $a_n < 0$ for all $n \geq 1$, hence a_n is bounded. We conclude by the Monotone Convergence Theorem that a_n converges. $\diamond$

Example 43. Determine whether the sequence $a_n = \cos(\pi n)$ converges. If so, find the limit.

Solution. Considering the graph of $\cos(\pi x)$, we suspect that a_n diverges. We will prove this by establishing two convergent subsequences of a_n with different limits. Observe that $\cos(2n\pi) = 1$ and $\cos((2n+1)\pi) = -1$ for all integers $n \geq 1$, hence we have that $\lim_{n\to\infty} a_{2n} = 1$ and $\lim_{n\to\infty} a_{2n+1} = -1$. $\diamond$

Basics of Infinite Series

Example 44. Find the first five partial sums of the infinite series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$.

Solution. We compute each partial sum in turn.

$$a_1 = \sum_{k=1}^{1} \frac{1}{k} = \frac{1}{1} = 1$$

$$a_2 = \sum_{k=1}^{2} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$a_3 = \sum_{k=1}^{3} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{6+3+2}{6} = \frac{11}{6}$$

$$a_4 = \sum_{k=1}^{4} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{12+6+4+3}{12} = \frac{25}{12}$$

$$a_5 = \sum_{k=1}^{5} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{60+30+20+15+12}{60} = \frac{137}{60}$$

◇

Example 45. Find an explicit formula for the partial sums of the infinite series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k}$; then, determine whether the infinite series converges. If so, find the value of the infinite series.

Solution. Observe that the nth partial sum of $\sum_{k=1}^{\infty} \frac{1}{2^k}$ is given by

$$s_n = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = \frac{2^{n-1} + 2^{n-2} + \cdots + 1}{2^n} = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

Considering that $s_n = 1 - \frac{1}{2^n}$ is the difference of a constant sequence and a convergent geometric sequence, we conclude that s_n converges, hence the infinite series converges with value

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1.$$ ◇

Example 46. Determine if the infinite series $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{2k} - \frac{1}{2k+2}\right)$ converges. If so, find its value.

Solution. Observe that the nth partial sum of the infinite series is given by

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{2k} - \frac{1}{2k+2} \right) = \frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{2n+2}.$$

Consequently, each of the negative summands cancels with the subsequent positive summand, and we find that this infinite series is telescoping with $c = \frac{1}{2}$ and $f(n) = -\frac{1}{2n+2}$. We conclude that

$$\sum_{k=1}^{\infty} \left(\frac{1}{2k} - \frac{1}{2k+2} \right) = \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{1}{2k} - \frac{1}{2k+2} \right) = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(\frac{1}{2} - \frac{1}{2n+2} \right) = \frac{1}{2}. \qquad \diamond$$

Example 47. Determine if the infinite series $\displaystyle\sum_{n=1}^{\infty} (\ln e^{\pi})^n$ converges. If so, find its value.

Solution. Considering that $\ln e^{\pi} = \pi \ln e = \pi > 1$, the sum diverges. $\diamond$

Example 48. Determine if the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$ converges. If so, find its value.

Solution. By Example 36, we have that

$$\lim_{n \to \infty} \frac{n!}{10^n} = \lim_{n \to \infty} \frac{1}{\frac{10^n}{n!}} = \infty \neq 0,$$

hence the series diverges by the Divergence Test. $\diamond$

Example 49. Determine if the infinite series $\displaystyle\sum_{n=7}^{\infty} \frac{n^3+n^2+n+1}{n^3-n^2+n-1}$ converges. If so, find its value.

Solution. Observe that

$$\lim_{n \to \infty} \frac{n^3+n^2+n+1}{n^3-n^2+n-1} = \lim_{n \to \infty} \frac{n^3\left(1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}\right)}{n^3\left(1-\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}\right)} = \lim_{n \to \infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}}{1-\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}} = 1 \neq 0,$$

hence the series diverges by the Divergence Test. $\diamond$

Caution. Often, upon first learning the Divergence Theorem, students get mixed up in the logic of what exactly the theorem guarantees. Put explicitly, the theorem says that

1.) if the limit of the sequence a_n of terms of the series *does not converge to* 0, then it is impossible for the series $\sum a_n$ to converge, and

2.) if the series $\sum a_n$ converges, then the sequence a_n of terms of the series *must converge to* 0.

Consequently, we are able to decipher when a series *diverges* by the Divergence Test — hence the name; however, the drawback is that we cannot tell that a series *converges* by the Divergence Test.

Example 50. Prove that $a_n = \dfrac{1}{\sqrt{n}}$ satisfies $\displaystyle\lim_{n \to \infty} a_n = 0$; then, prove that $\sum a_n$ diverges.

Solution. Observe that
$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0.$$
Considering that $\sqrt{n}$ is an increasing function, it follows that
$$s_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \underbrace{\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}}_{n \text{ summands}} = \frac{n}{\sqrt{n}} = \sqrt{n}.$$
Consequently, we have that $\sum a_n = \lim_{n\to\infty} s_n > \lim_{n\to\infty} \sqrt{n} = \infty$, and $\sum a_n$ diverges. $\diamond$

Convergence Tests for Series

Example 51. Use the Integral Test to prove that $\displaystyle\sum_{n=m}^{\infty} \frac{1}{n}$ diverges for any positive integer m.

Proof. Consider the function $f(x) = \frac{1}{x}$. Given a positive integer m, we have that $f(x) > 0$ for all $x \geq m$, so $f(x)$ is positive. Observe that $f'(x) = -\frac{1}{x^2} < 0$, hence $f(x)$ is decreasing for all real numbers $x \neq 0$. Last, $f(x)$ is the quotient of a continuous function by a continuous function, hence $f(x)$ is continuous. Consequently, we may use the Integral Test. We have that
$$\int_m^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_m^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|x| \bigg|_m^b = \lim_{b\to\infty} [\ln(b) - \ln(m)] = \infty.$$
Considering that $\int_m^\infty \frac{1}{x}\,dx$ diverges, it follows that $\sum_{n=m}^\infty \frac{1}{n}$ diverges by the Integral Test. $\square$

Example 52. Use the Integral Test to determine the convergence of $\displaystyle\sum_{n=m}^{\infty} \frac{1}{1+n^2}$.

Solution. Consider the function $f(x) = \frac{1}{1+x^2}$. Given a positive integer m, we have that $f(x) > 0$ for all $x \geq m$, so $f(x)$ is positive. Observe that $f'(x) = -\frac{2x}{(1+x^2)^2} < 0$ for all $x > 0$, hence $f(x)$ is decreasing for all $x > 0$. Last, $f(x)$ is the quotient of a continuous function by a continuous function, hence $f(x)$ is continuous. Consequently, we may use the Integral Test. We have that
$$\int_m^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_m^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \arctan x \bigg|_m^b = \lim_{b\to\infty} [\arctan(b) - \arctan(m)] = \frac{\pi}{4} - \arctan(m).$$
Considering that $\int_m^\infty \frac{1}{1+x^2}\,dx$ converges, it follows that $\sum_{n=m}^\infty \frac{1}{1+n^2}$ converges by the Integral Test. $\diamond$

Example 53. Use the p-Series Test to determine the convergence of $\displaystyle\sum_{n=m}^{\infty} \frac{1}{\sqrt[5]{n^7}}$.

Solution. Observe that $\sqrt[5]{n^7} = n^{7/5}$. Considering that $\frac{7}{5} > 1$, this series converges. $\diamond$

Example 54. Use the p-Series Test to determine the convergence of $\displaystyle\sum_{n=m}^{\infty} \frac{1}{\sqrt[7]{n^5}}$.

Solution. Observe that $\sqrt[7]{n^5} = n^{5/7}$. Considering that $\frac{5}{7} \leq 1$, this series diverges. $\diamond$

Example 55. Use the Direct Comparison Test to determine the convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{1+n^5}$.

Solution. Considering the exponents in the numerator and denominator, we suspect that the series converges: it will (eventually) behave like the convergent p-series $\frac{1}{n^3}$ because for sufficiently large n, we have that $1 + n^5 \approx n^5$ so that $\frac{n^2}{1+n^5} \approx \frac{n^2}{n^5} = \frac{1}{n^3}$. We proceed as follows.

$$n^5 \leq 1 + n^5 \implies \frac{1}{n^5} \geq \frac{1}{1+n^5} \implies \frac{1}{n^3} = \frac{n^2}{n^5} \geq \frac{n^2}{1+n^5}$$

Considering that $0 \leq \frac{n^2}{1+n^5} \leq \frac{1}{n^3}$ for all integer $n \geq 1$, the series converges by Direct Comparison. $\diamond$

Example 56. Use the Direct Comparison Test to determine the convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt[7]{n^5+1}}$.

Solution. We will establish that there exists a real number M with $0 \leq \frac{1}{n} \leq \frac{1}{\sqrt[7]{n^5+1}}$ for all integers $n > M$. Observe that this is equivalent to showing that $n \geq \sqrt[7]{n^5+1}$ or $n^7 - n^5 - 1 \geq 0$. Considering that $\lim_{n\to\infty}(n^7 - n^5 - 1) = \infty$ (look at the end behavior of its graph), it follows that there exists a real number M sufficiently large such that $n^7 - n^5 - 1 \geq 0$ for all integers $n > M$. We have therefore established that $0 \leq \frac{1}{n} \leq \frac{1}{\sqrt[7]{n^5+1}}$, hence the series diverges by the Direct Comparison Test. $\diamond$

Example 57. Use the Limit Comparison Test to determine the convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt[7]{n^5+1}}$.

Solution. Our initial urge is to use the Direct Comparison Test with $a_n = \frac{1}{\sqrt[7]{n^5}} = \frac{1}{n^{5/7}}$; however,

$$\sqrt[7]{n^5} \leq \sqrt[7]{n^5+1} \implies \frac{1}{\sqrt[7]{n^5}} \geq \frac{1}{\sqrt[7]{n^5+1}},$$

hence the inequality is going the wrong direction, and the Direct Comparison Test fails with this a_n. We have seen in Example 56 that we can directly compare with $a_n = \frac{1}{n}$, but we can salvage our original idea and use the Limit Comparison Test with $b_n = \frac{1}{\sqrt[7]{n^5}}$ to obtain something fruitful.

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[7]{n^5+1}}}{\frac{1}{\sqrt[7]{n^5}}} = \lim_{n\to\infty} \frac{\sqrt[7]{n^5}}{\sqrt[7]{n^5+1}} = \lim_{n\to\infty} \sqrt[7]{\frac{n^5}{n^5+1}} = \sqrt[7]{\lim_{n\to\infty} \frac{n^5}{n^5+1}} = \sqrt[7]{\lim_{n\to\infty} \frac{1}{1+\frac{1}{n^5}}} = 1$$

By the Limit Comparison Test, the series in question converges if and only if the p-series with $p = \frac{5}{7} \leq 1$ converges. Considering that this p-series diverges, the series in question diverges. $\diamond$

Example 58. Use the Limit Comparison Test to determine the convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n^3 - n^2 + n - 1}{n^4 - n^3 + n^2 - n + 1}$.

Solution. Considering the end behavior of the polynomials in the numerator and denominator, we have that $n^3 - n^2 + n - 1 \approx n^3$ and $n^4 - n^3 + n^2 - n + 1 \approx n^4$ for sufficiently large n, hence the terms of the series eventually behave like $\frac{n^3}{n^4} = \frac{1}{n}$. Put another way, we have that

$$\lim_{n\to\infty} \frac{\frac{n^3-n^2+n-1}{n^4-n^3+n^2-n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^4 - n^3 + n^2 - n}{n^4 - n^3 + n^2 - n + 1} = \lim_{n\to\infty} \frac{1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3}}{1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \frac{1}{n^4}} = 1.$$

By the Limit Comparison Test, the series in question converges if and only if the p-series with $p = 1 \leq 1$ converges. Considering that this p-series diverges, the series in question diverges. $\diamond$

Absolute and Conditional Convergence

Example 59. Determine if the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n n^{-\pi}$ is absolutely convergent.

Solution. Considering that $|(-1)^n n^{-\pi}| = n^{-\pi} = \frac{1}{n^\pi}$ gives rise to a convergent p-series ($p = \pi > 1$), we conclude that the series converges absolutely. $\diamond$

Example 60. Determine if the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n n^{-2}$ converges.

Solution. Considering that $|(-1)^n n^{-2}| = n^{-2} = \frac{1}{n^2}$ gives rise to a convergent p-series ($p = 2 > 1$), we conclude that the series converges absolutely. By the above fact, the series converges. $\diamond$

Example 61. Prove that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ converges.

Solution. Observe that the sequence $b_n = \frac{1}{n}$ is $\underline{\text{positive}}$ for all $n \geq 1$. Further, we have that $n+1 \geq n$ so that $\frac{1}{n+1} \leq \frac{1}{n}$, hence b_n is $\underline{\text{decreasing}}$. Considering that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, we conclude that the alternating harmonic series converges by the Alternating Series Test. $\diamond$

Example 62. Determine all values of p such that the alternating p-series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^p}$ converges. Explain how this differs from the case of the non-alternating p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Solution. We note that if $p = 0$, then $(-1)^n \frac{1}{n^p} = (-1)^n$, and the series $\sum_{n=1}^{\infty}(-1)^n$ diverges because its sequence of partial sums $s_{2k+1} = -1$ and $s_{2k} = 0$ oscillates. Likewise, if $p < 0$, then we have that $-p > 0$ so that $(-1)^n \frac{1}{n^p} = (-1)^n n^{-p}$ grows arbitrarily large in absolute value. By the Divergence Test, therefore, the series $\sum_{n=1}^{\infty}(-1)^n n^{-p}$ diverges. On the other hand, if $p > 0$, then

$$\frac{d}{dx}\frac{1}{x^p} = -\frac{p}{x^{p+1}} < 0$$

for all $x > 0$, hence $b_n = \frac{1}{n^p}$ is $\underline{\text{positive}}$ and $\underline{\text{decreasing}}$. Considering that $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$ for all $p > 0$, we conclude by the Alternating Series Test that $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^p}$ converges for $p > 0$.

We note that the non-alternating p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$, so the alternating p-series converge for all positive p-values, and both diverge for all negative p-values. $\diamond$

The Ratio Test

Example 63. Use the Ratio Test to determine if the series $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}$ converges.

Solution. We have that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\cdot\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^{n+1}}{(n+1)!}\cdot\frac{n!}{e^n}\right| = \lim_{n\to\infty}\frac{e^{n+1}}{e^n}\cdot\frac{n!}{(n+1)n!} = \lim_{n\to\infty}\frac{e}{n+1} = 0 < 1.$$

By the Ratio Test, we conclude that $\sum_{n=0}^{\infty}\frac{e^n}{n!}$ converges absolutely, hence it converges. ◇

Example 64. Use the Ratio Test to determine if the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{(n^2)!}$ converges.

Solution. We have that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\cdot\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{((n+1)^2)!}\cdot\frac{(n^2)!}{n^n}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n}\cdot\frac{(n^2)!}{((n+1)^2)!} \\
&= \lim_{n\to\infty}\frac{(n+1)(n+1)^n}{n^n}\cdot\frac{(n^2)!}{(n^2+n+1)!} \\
&= \lim_{n\to\infty}(n+1)\left(\frac{n+1}{n}\right)^n\cdot\frac{(n^2)!}{(n^2+n+1)(n^2+n)\cdots(n^2+1)(n^2)!} \\
&= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\frac{n+1}{(n^2+n+1)(n^2+n)\cdots(n^2+1)} \\
&= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\lim_{n\to\infty}\frac{n+1}{(n^2+n+1)(n^2+n)\cdots(n^2+1)} \\
&= e\cdot 0 = 0.
\end{aligned}$$

By the Ratio Test, we conclude that $\sum_{n=0}^{\infty}\frac{n^n}{(n^2)!}$ converges absolutely, hence it converges. ◇

Power Series and Taylor Series

Example 65. Prove that the power series $\displaystyle\sum_{k=0}^{\infty} kx^k$ converges for $x=\frac{1}{2}$ and diverges for $x=1$.

Solution. Observe that the series $\sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k$ converges absolutely by the Ratio Test:

$$\lim_{k\to\infty}\left|\frac{(k+1)\left(\frac{1}{2}\right)^{k+1}}{k\left(\frac{1}{2}\right)^k}\right| = \frac{1}{2}\cdot\lim_{k\to\infty}\frac{k+1}{k} \overset{\text{L'H}}{=} \frac{1}{2}\cdot\lim_{k\to\infty}\frac{1}{1} = \frac{1}{2} < 1.$$

On the other hand, for $x=1$, we have that $\sum_{k=0}^{\infty} k$ diverges by the Divergence Test as $\lim\limits_{k\to\infty} k \neq 0$. ◇

Example 66. Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution. We proceed by the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} \right| && \text{(Group like terms.)} \\
&= |x| \lim_{n\to\infty} \frac{n!}{(n+1)!} && \text{(Cancel, and pull out constants.)} \\
&= |x| \lim_{n\to\infty} \frac{n!}{(n+1)n!} && \text{(Write } n! \text{ as a factor of } (n+1)!\text{.)} \\
&= |x| \lim_{n\to\infty} \frac{1}{n+1} && \text{(Cancel common factors.)} \\
&= 0.
\end{aligned}$$

We conclude that regardless of the value of x, the power series in question converges. Consequently, the radius of convergence is $R = \infty$ and the interval of convergence is $I = (-\infty, \infty)$. ◇

Example 67. Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{3^n}$.

Solution. Observe that we can view this power series as a geometric series with common ratio

$$r = \frac{x-3}{-3}.$$

We conclude by the Geometric Series Test that the power series converges if and only if $|r| < 1$. Consequently, the radius of convergence is $R = 3$ and the interval of convergence is $I = (0, 6))$. ◇

Example 68. Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} n^n x^n$.

Solution. We proceed by the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1} x^{n+1}}{n^n x^n} \right| &= |x| \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{n^n} \\
&= |x| \lim_{n\to\infty} (n+1) \left(1 + \frac{1}{n}\right)^n \\
&= |x| \lim_{n\to\infty} (n+1) \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \infty.
\end{aligned}$$

We conclude that the power series diverges for all $x \neq 0$, i.e., $R = 0$ and $I = \{0\}$. $\diamond$

Example 69. Use the geometric series to find a power series identity for the following functions; then, state the radius and interval of convergence for each power series.

(a.) $\dfrac{1}{1-x}$

(b.) $\dfrac{1}{1+x}$

(c.) $\dfrac{1}{1+x^2}$

Solution. (a.) Observe that for all real numbers x such that $|x| < 1$, this is the sum of a convergent geometric series with $c = 1$ and $r = x$. Consequently, we have that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

By the Geometric Series Test, this series diverges whenever $|x| \geq 1$, hence the radius of convergence is $R = 1$ and the interval of convergence is $I = (-1, 1)$. $\diamond$

Solution. (b.) Observe that for all real numbers x such that $|x| = |-x| < 1$, this is the sum of a convergent geometric series with $c = 1$ and $r = -x$. Consequently, we have that

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n.$$

By the Geometric Series Test, this series diverges whenever $|-x| = |x| \geq 1$, hence the radius of convergence is $R = 1$ and the interval of convergence is $I = (-1, 1)$. $\diamond$

Solution. (c.) Observe that for all real numbers x such that $|-x^2| < 1$, this is the sum of a convergent geometric series with $c = 1$ and $r = -x^2$. Consequently, we have that

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

By the Geometric Series Test, this series diverges whenever $x^2 = |-x^2| \geq 1$, hence the radius of convergence is $R = 1$ and the interval of convergence is $I = (-1, 1)$. $\diamond$

Example 70. Use Example 69 to find a power series identity for the following functions; then, state the radius and interval of convergence for each power series.

(a.) $\dfrac{1}{(1-x)^2}$

(b.) $\ln|1 + x|$

(c.) $\arctan x$

Solution. (a.) Observe that

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2}(-1) = (1-x)^{-2} = \frac{1}{(1-x)^2}.$$

Using Example 69(a.) above, it follows that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \sum_{n=1}^{\infty} n x^{n-1}.$$

We do not change the radius or interval of convergence when differentiating power series, hence we have that the radius of convergence is $R = 1$ and the interval of convergence is $I = (-1, 1)$. ◇

Solution. (b.) Observe that

$$\ln|1+x| = \int \frac{1}{1+x}\,dx.$$

Using Example 69(b.) above, it follows that

$$\ln|1+x| + C = \int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx = \sum_{n=0}^{\infty} (-1)^n \int x^n\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$

By plugging in $x = 0$ and using the fact that $\ln(1) = 0$, it follows that $C = 0$. We do not change the radius or interval of convergence when antidifferentiating power series, hence we have that the radius of convergence is $R = 1$ and the interval of convergence is $I = (-1, 1)$. ◇

Solution. (c.) Observe that

$$\arctan x = \int \frac{1}{1+x^2}\,dx.$$

Using Example 69(c.) above, it follows that

$$\arctan x = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

By plugging in $x = 0$ and using the fact that $\arctan(0) = 0$, it follows that $C = 0$. We do not change the radius or interval of convergence when antidifferentiating power series, hence we have that the radius of convergence is $R = 1$ and the interval of convergence is $I = (-1, 1)$. ◇

Example 71. Prove that $T_n(x)$ is an nth-order approximation of $f(x)$ at $x = c$.

Solution. By definition, we must show that $T_n(c) = f(c)$, $T_n'(c) = f'(c)$, $T_n''(c) = f''(c)$, and in general, the kth derivative of $T_n(x)$ evaluated at $x = c$ is equal to kth derivative of $f(x)$ evaluated at $x = c$ for all integers $0 \leq k \leq n$. Explicitly, we must show that

$$\frac{d^k}{dx^k} T_n(x)\bigg|_{x=c} = \frac{d^k}{dx^k} f(x)\bigg|_{x=c}$$

for all integers $0 \leq k \leq n$. Observe that we have that

$$\frac{d}{dx}T_n(x) = f'(c) + \frac{f''(c)}{2!} \cdot 2(x-c) + \frac{f'''(c)}{3!} \cdot 3(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!} \cdot n(x-c)^{n-1}$$

$$= f'(c) + f''(c)(x-c) + \frac{f'''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{(n-1)!}(x-c)^{n-1},$$

$$\frac{d^2}{d^2x}T_n(x) = f''(c) + \frac{f'''(c)}{2!} \cdot 2(x-c) + \cdots + \frac{f^{(n)}(c)}{(n-1)!} \cdot (n-1)(x-c)^{n-2}$$

$$= f''(c) + f'''(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{(n-2)!}(x-c)^{n-2},$$

and in general, for all integers $0 \leq k \leq n$, we have that

$$\frac{d^k}{d^kx}T_n(x) = f^{(k)}(c) + f^{(k+1)}(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{(n-k)!}(x-c)^{n-k},$$

from which it follows that $\displaystyle \frac{d^k}{dx^k}T_n(x)\bigg|_{x=c} = f^{(k)}(c) = \frac{d^k}{dx^k}f(x)\bigg|_{x=c}$ as desired. ◇

Example 72. Give a closed form for the sequence $a_n = f^{(n)}(x)$ of derivatives of $f(x) = e^x$. Use this to find the nth Taylor polynomial $T_n(x)$ of e^x centered at $x = 0$.

Solution. Observe that $\frac{d}{dx}e^x = e^x$ so that $\frac{d^n}{d^nx}e^x = e^x$ for all integers $n \geq 0$. We have therefore that $a_n = f^{(n)}(x) = e^x$ so that $f^{(n)}(0) = 1$ for each integer $n \geq 0$. Our Taylor polynomial is therefore

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}.$$ ◇

Example 73. Give a closed form for the sequence $a_n = f^{(n)}(x)$ of derivatives of $f(x) = \cos x$. Use this to find the nth Taylor polynomial $T_n(x)$ of $\cos x$ centered at $x = 0$.

Solution. Observe that we have that

$$\begin{aligned} f'(x) &= -\sin x, \\ f''(x) &= -\cos x, \\ f'''(x) &= \sin x, \text{ and} \\ f^{(4)}(x) &= \cos x = f(x). \end{aligned}$$

Consequently, our sequence of derivatives is given by

$$\begin{aligned} f^{(4k)}(x) &= \cos x, \\ f^{(4k+1)}(x) &= -\sin x, \\ f^{(4k+2)}(x) &= -\cos x, \text{ and} \\ f^{(4k+3)}(x) &= \sin x \end{aligned}$$

for all integers $k \geq 0$, from which it follows that

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = 1 + 0 \cdot x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 + \frac{-1}{6!}x^6 + \frac{0}{7!}x^7 + \cdots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots = \sum_{k=0}^{(n-1)/2} \frac{(-1)^k x^{2k}}{(2k)!}. \qquad \diamond$$

Example 74. Use the Error Bound Theorem to find the maximum error in approximating e^2 with $f(x) = e^x$ and the fourth Taylor polynomial $T_4(x)$ centered at $x = 0$.

Solution. We will use the Error Bound Theorem with $c = 0$, $x = 2$, and $n = 4$. Considering that $f^{(n)}(x) = e^x$ for all integers $n \geq 0$, it follows that $f^{(5)}(x)$ exists and is continuous. Further, we have that $f^{(5)}(x) = e^x$ is increasing so that $|f^{(5)}(u)| \leq e^2 = f^{(5)}(2) = \max\{f^{(5)}(u) \,|\, 0 \leq u \leq 2\}$. By the Error Bound Theorem, it follows that the maximum error in approximating e^2 with $f(x) = e^x$ and the fourth Taylor polynomial $T_4(x)$ centered at $x = 0$ is given by

$$K \cdot \frac{|x-c|^{n+1}}{(n+1)!} = e^2 \cdot \frac{|2-0|^5}{5!} = \frac{32e^2}{120}. \qquad \diamond$$

Example 75. Use the Error Bound Theorem to find an integer $n \geq 0$ such that

$$|\cos(1) - T_n(1)| \leq \frac{1}{1000}.$$

Solution. Considering that all the derivatives of $f(x) = \cos x$ are differentiable (and hence continuous), it follows that we may invoke the Error Bound Theorem. Explicitly, recall that the derivatives of $\cos x$ are $\pm \sin x$ and $\pm \cos x$, hence we have that $|f^{(n)}(x)| \leq 1$ since $|\pm \sin x| \leq 1$ and $|\pm \cos x| \leq 1$. By Example 73, we may use the nth Taylor polynomial of $\cos x$ centered at $c = 0$. By the Error Bound Theorem with $x = 1$, $c = 0$, and $K = 0$, we have that

$$1 \cdot \frac{1}{(n+1)!} = K \cdot \frac{|1-0|^{n+1}}{(n+1)!} \leq \frac{1}{1000} \iff 1000 \leq (n+1)!.$$

Considering that $1000 \leq 7! = 5040$ and $1000 > 6! = 840$, we may take $n = 6$. $\diamond$

Example 76. Use Example 72 to find the Maclaurin series for $f(x) = e^x$.

Solution. By Example 72, the Taylor polynomial for e^x centered at $x = 0$ is given by

$$T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!},$$

from which it follows that the Maclaurin series for e^x is given by

$$T(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \qquad \diamond$$

Example 77. Use Example 73 to find the Maclaurin series for $f(x) = \cos x$.

Solution. By Example 73, the Taylor polynomial for $\cos^x$ centered at $x = 0$ is given by

$$T_n(x) = \sum_{k=0}^{(n-1)/2} \frac{(-1)^k x^{2k}}{(2k)!},$$

from which it follows that the Maclaurin series for $\cos x$ is given by

$$T(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}. \qquad \diamond$$

Example 78. Use Example 77 to find the Maclaurin series for $f(x) = \sin x$.

Solution. By Example 77, the Maclaurin series for $\cos x$ is given by

$$t(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

Considering that $\cos x = \frac{d}{dx} \sin x$, we can simply integrate the Maclaurin series for $\cos x$ to obtain the Maclaurin series for $\sin x$. Consequently, we have that

$$T(x) + C = \int t(x)\, dx = \int \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int x^{2k}\, dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

By plugging in $x = 0$, we find that $T(0) + C = 0$. Using the fact that $T(0) = \sin(0) = 0$ by construction, we conclude that $C = 0$ so that the Maclaurin series for $\sin x$ is given by

$$T(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}. \qquad \diamond$$

Example 79. Use Examples 76, 77, and 78 in addition to the above fact to find the power series expansions of e^x, $\cos x$, and $\sin x$. Determine the radius and interval of convergence for each of these.

Solution. Given a real number R, by the previous examples, we have that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \text{ and}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

since each of these functions is continuously differentiable for each integer $n \geq 0$ on any open interval $(-R, R)$. By the Ratio Test, each of these converges for all real numbers x:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = |x| \cdot \lim_{n\to\infty} \frac{n!}{(n+1)n!} = |x| \cdot \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

hence the power series expansion for e^x converges for all x (and similarly for $\cos x$ and $\sin x$). $\diamond$

Example 80. Use Example 79 to find the Taylor series of the following; then, state their centers.

(a.) $f(x) = x^3 \cos x$

(b.) $g(x) = e^{1-x^2}$

(c.) $h(x) = e^{x-4}$

(d.) $k(x) = \dfrac{x - \sin x}{x}$

Solution. (a.) By Example 79, we have that

$$x^3 \cos x = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n)!}$$

is the power series expansion of $x^3 \cos x$ centered at $c = 0$. $\diamond$

Solution. (b.) By Example 79, we have that

$$e^{1-x^2} = e \cdot e^{-x^2} = e \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n e x^{2n}}{n!}$$

is the power series expansion of e^{1-x^2} centered at $c = 0$. $\diamond$

Solution. (c.) By Example 79, we have that

$$e^{x-4} = \frac{e^x}{e^4} = \frac{1}{e^4} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{e^4 n!}$$

is the power series expansion of e^{x-4} centered at $c = 0$. We could have also found that

$$e^{x-4} = \sum_{n=0}^{\infty} \frac{(x-4)^n}{n!}$$

is the power series expansion of e^{x-4} centered at $c = 4$. $\diamond$

Solution. (d.) By Example 79, we have that

$$x - \sin x = x - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$$

is the power series expansion of $x - \sin x$ centered at $c = 0$. Consequently, we have that

$$\frac{x - \sin x}{x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n+1)!}$$

is the power series expansion of $\frac{x - \sin x}{x}$ centered at $c = 0$. $\diamond$

Example 81. Verify that L'Hôpital's Rule can be used to compute the limit

$$\lim_{x \to 0} \frac{x - \sin x}{x^3 \cos x};$$

then, explain the difficulty in doing so. Ultimately, compute the limit using power series.

Solution. By plugging in $x = 0$ to the numerator and denominator, we find that the form of the limit is $0/0$. We can therefore implement L'Hôpital's Rule to find that

$$\lim_{x \to 0} \frac{x - \sin x}{x^3 \cos x} \stackrel{\text{L'H}}{=} \lim_{x \to 0} \frac{1 - \cos x}{-x^3 \sin x + 3x^2 \cos x}.$$

Once again, we plug in $x = 0$ to the numerator and denominator to find that the form of the limit is $0/0$, and we can implement L'Hôpital's Rule again; however, we will have to use the Product Rule twice. Ultimately, we find that in order to differentiate the denominator, we have to use the Product Rule twice more than the last time, and we always have a $\pm x^3 \sin x$ or $\pm x^3 \cos x$ term.

By Example 80(a.) and (d.), we may use the power series expansions of $x - \sin x$ and $x^3 \cos x$ centered at $c = 0$ to compute the limit. Explicitly, we have that

$$\lim_{x \to 0} \frac{x - \sin x}{x^3 \cos x} = \lim_{x \to 0} \frac{\frac{x^3}{3} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots}{x^3 - \frac{x^5}{2!} + \frac{x^7}{4!} - \cdots}$$

$$= \lim_{x \to 0} \frac{x^3\left(\frac{1}{3} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots\right)}{x^3\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots} = \frac{1}{3}.$$

◇

Example 82. Explain the difficulty in trying to find the antiderivative of $\sin(x^2)$; then, compute the power series expansion of the antiderivative $\sin(x^2)$, and state its radius of convergence.

Solution. We would like to compute $\int \sin(x^2)\,dx$. Our first guess would be to try to use substitution to find the antiderivative. Of course, we would let $u = x^2$ so that $du = 2x\,dx$. But this gets us nowhere because we do not have a factor of $2x$ in the integrand. Our second guess is to use $u = \sin(x)^2$ so that $du = \cos(x^2)(2x)\,dx$, but this is also useless. By Example 79, we have that

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

is the power series expansion of $\sin(x^2)$ centered at $c = 0$. Consequently, we have that

$$S(x) + C = \int \sin(x^2)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{4n+2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!},$$

where $S(x)$ is an antiderivative of $\sin(x^2)$. Considering that antidifferentiation does not change the radius of convergence of a power series, the radius of convergence of this power series is $R = \infty$. ◇

Example 83. Explain the difficulty in trying to find the antiderivative of e^{1-x^2}; then, compute the power series expansion of the antiderivative e^{1-x^2}, and state its radius of convergence.

Solution. We would like to compute $\int e^{1-x^2}\,dx$. Our first guess would be to try to use substitution to find the antiderivative. Of course, we would let $u = x^2$ so that $du = 2x\,dx$. But this gets us nowhere because we do not have a factor of $2x$ in the integrand. Our second guess is to use $u = e^{1-x^2}$ so that $du = -2xe^{1-x^2}\,dx$, but this is also useless. By Example 80(b.), we have that

$$e^{1-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n e x^{2n}}{n!}$$

is the power series expansion of e^{1-x^2} centered at $c = 0$. Consequently, we have that

$$F(x) + C = \int e^{1-x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n e x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n e}{n!} \int x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n e x^{2n+1}}{(2n+1)n!},$$

where $F(x)$ is an antiderivative of e^{1-x^2}. Considering that antidifferentiation does not change the radius of convergence of a power series, the radius of convergence of this power series is $R = \infty$. ◇

The Area Between Curves

Example 84. Compute the area of the region $\mathcal{R}$ cut out by the curves $f(x) = -x^2 + 4$ and $g(x) = x^2 - 4$ for all $-2 \leq x \leq 2$.

Solution. Graphing the curves, we find that $f(x) \geq g(x)$ for all $-2 \leq x \leq 2$. Further, observe that $-g(x) = f(x)$, from which it follows that $f(x) - g(x) = 2f(x)$. Consequently, we have that

$$\begin{aligned}
\text{area}(\mathcal{R}) &= \int_{-2}^{2} [f(x) - g(x)]\,dx \\
&= 2\int_{-2}^{2} f(x)\,dx && \text{(Use our above observation.)} \\
&= 4\int_{0}^{2} f(x)\,dx && \left(\textstyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(x) \text{ is an even function.}\right) \\
&= 4\int_{0}^{2} (-x^2 + 4)\,dx = 4\left[\frac{-x^3}{3} + 4x\right]_0^2 = 4\left(8 - \frac{8}{3}\right).
\end{aligned}$$

◇

Example 85. Compute the area of the region $\mathcal{R}$ cut out by the curves $f(x) = 2x + 1$ and $g(x) = 2x - 4$ for all $-1 \leq x \leq 2$. Explain how one can use geometry to verify that this area is correct. Last, discuss what would happen if we were not given values of a and b such that $a \leq x \leq b$.

Solution. Considering that $-4 < 1$, it follows that $2x - 4 \leq 2x + 1$ for all $-1 \leq x \leq 2$ so that

$$\text{area}(\mathcal{R}) = \int_{-1}^{2} (2x + 1 - 2x + 4)\, dx = \int_{-1}^{2} 5\, dx = 5(2 + 1) = 15.$$

Graphing the curves $f(x)$ and $g(x)$, we find that the region $\mathcal{R}$ is a rhombus with a base of 5 and a height of 3. Consequently, from elementary geometry, we have that $\text{area}(\mathcal{R}) = 5 \cdot 3 = 15$.

Last, if we were not given values of a and b such that $a \leq x \leq b$, the region $\mathcal{R}$ would have infinite area. Explicitly, the curves $f(x)$ and $g(x)$ form a pair of parallel lines, hence there does exist any real number c such that $f(c) = g(c)$, and the region $\mathcal{R}$ is not closed. We would have that

$$\text{area}(\mathcal{R}) = \int_{-\infty}^{\infty} (2x + 1 - 2x + 4)\, dx = \int_{-\infty}^{\infty} 5\, dx = \infty. \qquad \diamond$$

Example 86. Compute the area of the region $\mathcal{R}$ cut out by the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$.

Solution. We must first find real numbers a and b such that $f(x) \geq g(x)$ or $g(x) \geq f(x)$ for all $a \leq x \leq b$. We accomplish this by checking when $f(x) = g(x)$. Observe that

$$\sqrt{x} = x^2 \iff x = x^4 \iff x^4 - x = 0 \iff x(x^3 - 1) = 0 \iff x = 0 \text{ or } x = 1.$$

By inspection, we have that $f(x) \geq g(x)$ for all $0 \leq x \leq 1$. Consequently, we have that

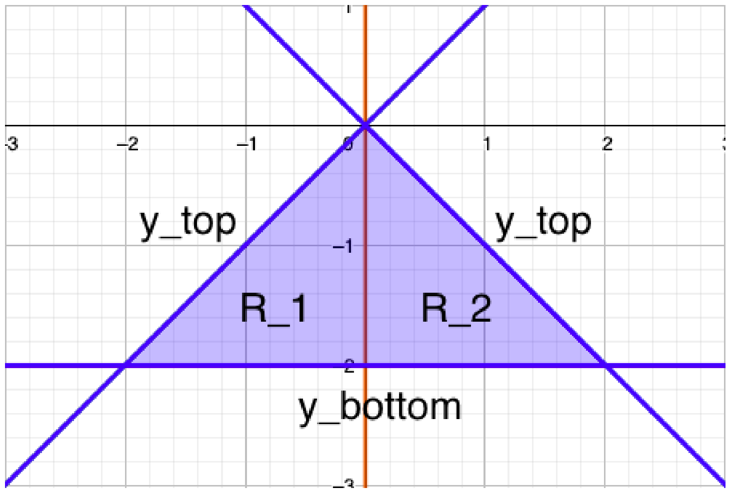
$$\text{area}(\mathcal{R}) = \int_0^1 [f(x) - g(x)]\, dx = \int_0^1 (\sqrt{x} - x^2)\, dx = \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}. \qquad \diamond$$

Example 87. Prove that the region $\mathcal{R}$ cut out by the curves $y = x$, $y = -x$, and $y = -2$ is not vertically simple; then, write the region $\mathcal{R}$ as the union of two vertically simple regions $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, and find the area of $\mathcal{R}$ by using the fact that $\text{area}(\mathcal{R}_1 \cup \mathcal{R}_2) = \text{area}(\mathcal{R}_1) + \text{area}(\mathcal{R}_2)$. Check that your final answer is correct using elementary geometry.

Solution. Observe that the curves $y = x$ and $y = -2$ intersect when $x = -2$. Likewise, the curves $y = -x$ and $y = -2$ intersect when $x = 2$. Consequently, we have that $a = -2$ and $b = 2$. On the contrary, if $\mathcal{R}$ were vertically simple, then there would exist well-defined curves y_{top} and y_{bottom} for all $-2 \leq x \leq 2$; however, for $-2 \leq x \leq 0$, we have that $y_{\text{top}} = x$ and $y_{\text{bottom}} = -2$, and for $0 \leq x \leq 2$, we have that $y_{\text{top}} = -x$ and $y_{\text{bottom}} = -2$. (One can check this graphically, too.) Consequently, the region $\mathcal{R}$ is not vertically simple; rather, we can write $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ with

$$\mathcal{R}_1 = \{(x, y) \mid -2 \leq x \leq 0 \text{ and } -2 \leq y \leq x\} \text{ and}$$
$$\mathcal{R}_2 = \{(x, y) \mid 0 \leq x \leq 2 \text{ and } -2 \leq y \leq -x\}.$$

Certainly, both of these regions are vertically simple. $\diamond$



Example 88. Prove that the region $\mathcal{R}$ of Example 87 is horizontally simple by exhibiting well-defined curves x_{left} and x_{right} for all $c \leq y \leq d$; then, compute the area of $\mathcal{R}$.

Solution. Graphing the curves, we find that $x_{\text{right}} = -y$ and $x_{\text{left}} = y$ for all $-2 \leq y \leq 0$. $\diamond$

Example 89. Describe the region in Example 86 as horizontally simple. List any observations you have about your description of the region; then, compute its area.

Solution. We must exhibit well-defined curves $x_{\text{right}} = g_2(y)$ and $x_{\text{left}} = g_1(y)$ for all $c \leq y \leq d$. Observe that $y = f(x) = \sqrt{x}$ implies that $x = y^2$. Likewise, we have that $y = g(x) = x^2$ implies that $x = \sqrt{y}$. Using Example 89, we have that $x_{\text{right}} = \sqrt{y}$ and $x_{\text{left}} = y^2$ for all $0 \leq y \leq 1$. But this is exactly the same description as in Example 86 with the names of x and y swapped, hence

$$\text{area}(\mathcal{R}) = \int_0^1 (x_{\text{right}} - x_{\text{left}})\, dy = \int_0^1 (\sqrt{y} - y^2)\, dy = \int_0^1 (\sqrt{x} - x^2)\, dx = \frac{1}{3}. \qquad \diamond$$

Example 90. Prove that the region $\mathcal{R}$ enclosed by the curves $y = x - 2$, $y = 2 - x$, $y = -x + 2$, and $y = -x - 2$ is neither vertically nor horizontally simple.

Proof. Graphing these curves, we find that $\mathcal{R}$ is a square that is standing on one of its corners. Consequently, the region $\mathcal{R}$ has symmetry about both the x- and y-axes, hence it suffices to check that $\mathcal{R}$ is not vertically simple. Considering that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ with

$$\mathcal{R}_1 = \{(x, y) \,|\, -2 \leq x \leq 0 \text{ and } -x - 2 \leq y \leq x + 2\} \text{ and}$$
$$\mathcal{R}_2 = \{(x, y) \,|\, 0 \leq x \leq 2 \text{ and } x - 2 \leq y \leq -x + 2\},$$

we conclude that $\mathcal{R}$ is not vertically simple, hence it is not horizontally simple. $\square$

Example 91. Compute $-\int_0^1 \ln x\, dx$ by viewing it as the area of some region $\mathcal{R}$.

Solution. Considering that $\ln(x) \leq 0$ for all $0 < x \leq 1$, it follows that $-\int_0^1 \ln x\, dx$ is the area of the region $\mathcal{R}$ bounded by the curves $y = 0$, $y = \ln x$, $x = 0$, and $x = 1$. Observe that $y = \ln x$ if and only if $x = e^y$. Further, we have that $d = \lim_{x \to 0^+} \ln x = -\infty$ and $c = \lim_{x \to 1^-} \ln x = 0$ so that

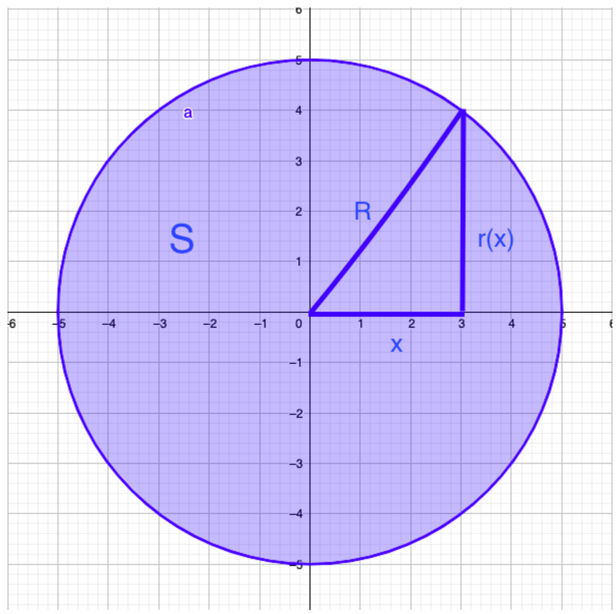
$$-\int_0^1 \ln x\, dx = \text{area}(\mathcal{R}) = \int_{-\infty}^0 (x_{\text{right}} - x_{\text{left}})\, dy = \lim_{a \to -\infty} \int_a^0 e^y\, dy = \lim_{a \to -\infty} e^y \Big|_a^0 = 1. \qquad \diamond$$

Volume, Density, and Average Value

Example 92. Prove that the volume of a sphere of radius $R > 0$ is $\frac{4}{3}\pi R^3$.

Solution. By the paragraph above, we can use the formula for the volume by cross-sectional area. Our cross sections may be taken vertically (i.e., perpendicular to the x-axis). Each of these is a circle of radius $r(x)$. By the Pythagorean Theorem, the diagram below gives that $[r(x)]^2 + x^2 = R^2$, from which it follows that $r(x) = \sqrt{R^2 - x^2}$. Considering that the vertical cross sections of a sphere are circles of radius $r(x)$, it follows that area$(\mathcal{S}) = \pi[r(x)]^2$. We conclude therefore that

$$\text{volume}(\mathcal{S}) = \int_{-R}^{R} \pi[r(x)]^2\,dx = \pi\int_{-R}^{R}(R^2 - x^2)\,dx = 2\pi\int_0^R (R^2 - x^2) = 2\pi\left[R^2x - \frac{x^3}{3}\right]_0^R = \frac{4}{3}\pi R^3. \diamond$$



Example 93. Prove that the volume of a right-circular cone of radius R and height H is $\frac{1}{3}\pi R^2 H$.

Solution. By the paragraph above, we can use the formula for the volume by cross-sectional area. Observe that the horizontal cross sections of a right-circular cone $\mathcal{C}$ of radius R and height H are circles of radius $r(y)$. Considering the similar triangles in the diagram below, we have that

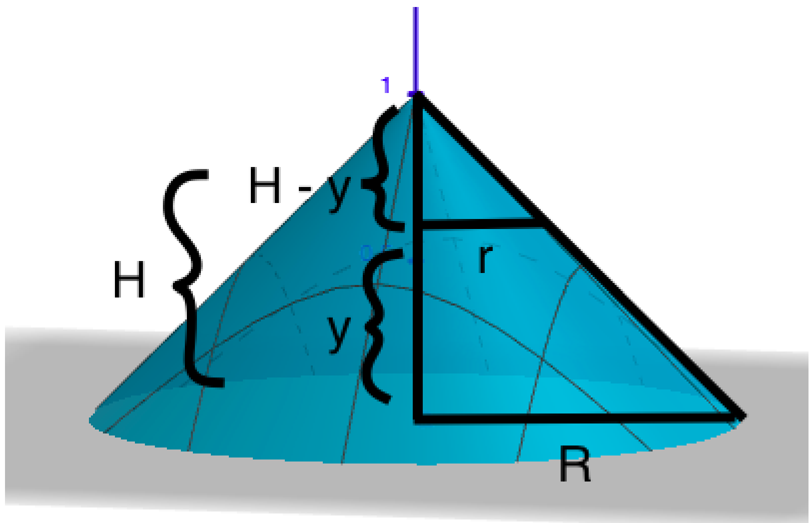
$$\frac{R}{H} = \frac{r(y)}{H - y},$$

from which it follows that $r(y) = \frac{R}{H}(H - y)$. Considering that the horizontal cross sections of a right-circular cone are circles of radius $r(y)$, it follows that $\text{area}(\mathcal{C}) = \pi[r(y)]^2$ so that

$$\text{volume}(\mathcal{C}) = \int_0^H \pi[r(y)]^2\,dy = \pi\frac{R^2}{H^2}\int_0^H (H-y)^2\,dy$$

$$= \pi\frac{R^2}{H^2}\int_0^H u^2\,du \qquad \text{(Use the substitution } u = H - y.\text{)}$$

$$= \pi\frac{R^2}{H^2}\frac{u^3}{3}\Big|_0^H = \frac{1}{3}\pi R^2 H. \qquad \diamond$$



Example 94. Compute the total mass of a rod of length 1 unit and lineal density $\rho(x) = xe^{x^2}$.

Solution. Considering that the length of the rod is 1 unit, we have that

$$\text{mass} = \int_0^1 \rho(x)\,dx = \int_0^1 xe^{x^2}\,dx = \frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}(e-1). \qquad \diamond$$

Example 95. Compute the average value of the function $f(x) = x^{-1}$ on the interval $\left[\frac{1}{e}, 1\right]$.

Solution. Considering that x^{-1} is continuous for all $x \neq 0$, it follows that x^{-1} is integrable on $\left[\frac{1}{e}, 1\right]$. We have therefore that

$$\text{average value of } x^{-1} \text{ on } \left[\frac{1}{e}, 1\right] = \frac{1}{1-\frac{1}{e}}\int_{1/e}^1 x^{-1}\,dx = \frac{1}{1-\frac{1}{e}}\ln x\Big|_{1/e}^1 = \frac{\ln(1) - \ln(1/e)}{1-\frac{1}{e}} = \frac{1}{1-\frac{1}{e}}. \qquad \diamond$$

Example 96. Consider a car travelling with a velocity of $v(t)$ units per minute. Prove that if the car enters a 325 unit-long tunnel at $t = 0$ minutes and exits at $t = 4$ minutes and the speed limit in the tunnel is 80 units per minute, then the car broke the speed limit at some point in the tunnel.

Solution. Observe that the position function $s(t)$ is the antiderivative of the velocity $v(t)$. Consequently, we have that $s(4) - s(0) = \int_0^4 v(t)\,dt$. By the Mean Value Theorem for Integrals, we conclude that there exists a real number $0 \leq c \leq 4$ such that

$$v(c) = \frac{1}{4-0}\int_0^4 v(t)\,dt = \frac{1}{4}[s(4) - s(0)] = \frac{1}{4} \cdot 325 > 80 \text{ units per minute.}$$

Our interpretation is that at some point in time $0 \leq c \leq 4$, the velocity of the car exceeded the speed limit of 80 units per minute, hence the car broke the speed limit at some point in the tunnel. $\diamond$