# MATH 126: Laboratory Workbook

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## Review of Limits and Continuity

Consider a function $f(x)$ defined on a subset $U$ of the real numbers $\mathbb{R}$. Given any value $a$ in $U$, we say that the **limit** of $f(x)$ as $x$ approaches $a$ (if it exists) is the quantity $L$ such that for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - L| < \varepsilon$. Put another way, the quantity $L$ can be made arbitrarily close to the value of $f(x)$ by taking $x$ to be sufficiently close in value to $a$. Conveniently, if the quantity $L$ exists, then we write $L = \lim_{x \to a} f(x)$.

**Example 1.** Compute the limit of $f(x) = x^2$ as $x$ approaches $a = 1$.

One-sided limits can be defined analogously to two-sided limits. Given that the left-hand limit of $f(x)$ as $x$ approaches $a$ exists, we write $L^- = \lim_{x \to a^-} f(x)$. Likewise, if the right-hand limit of $f(x)$ as $x$ approaches $a$ exists, we write $L^+ = \lim_{x \to a^+} f(x)$. Ultimately, the limit $L$ exists if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) \text{ or } L^- = L = L^+.$$

**Example 1, Revisited.** Compute the limit of $f(x) = x^2$ as $x$ approaches $a = 1$.

Given that $\lim_{x \to a} f(x) = f(a)$, we say that $f(x)$ is **continuous at** $a$. Further, if this is the case for every real number $a$ in $U$, then we say that $f(x)$ is **continuous on** $U$.

**Example 2.** Prove that the function $f(x) = |x|$ is continuous for all real numbers $a$.

Often, if a function is continuous for every real number in its domain $D$, we say that the function is **continuous**, by which we mean that $f(x)$ is continuous (in the above sense) on $D$. Graphically, we may detect that a function is continuous if we can draw it without lifting our pencil.

**Example 2, Revisited.** Prove that the function $f(x) = |x|$ is continuous for all real numbers $a$.

Continuous functions abound. Our prototypical examples of continuous functions are $x$, $e^x$, $\ln x$, $\sin x$, and $\cos x$. Further, the operations of addition, subtraction, multiplication, division, composition, and any finite combination of these preserve continuity (with some caveats).

# Review of Derivatives and L'Hôpital's Rule

Given a real number $h > 0$, consider the closed interval $[x, x+h]$. We define the secant line of $f(x)$ over this interval to be the line that passes through the points $(x, f(x))$ and $(x+h, f(x+h))$. Observe that the slope of the secant line is given by the difference quotient

$$D_x(h) = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}.$$

By taking the limit of $D_x(h)$ as $h$ approaches 0, we obtain the **derivative** of $f(x)$

$$\frac{d}{dx} f(x) \stackrel{\text{def}}{=} f'(x) \stackrel{\text{def}}{=} \lim_{h \to 0} D_x(h) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Of course, this limit might not always exist; however, when it does, we interpret it geometrically as the slope of the line tangent to $f(x)$ at the point $(x, f(x))$. Given that the quantity $f'(x)$ exists, we say that $f(x)$ is **differentiable**. One fundamental interpretation of the derivative in the context of a function that measures something physical (e.g., velocity) is as the instantaneous rate of change.

**Example 3.** Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = x^2$.

**Fact:** If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $a$. Put another way, a function that is differentiable at a point is necessarily continuous there. Conversely, there exists a function that is continuous at every point in its domain but not differentiable at every point in its domain.

*Proof.* We will assume that $f(x)$ is differentiable at $x = a$. Consequently, the limit

$$f'(a) = \lim_{h \to 0} D_a(h) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. Using the substitution $x = a + h$, it follows that $h = x - a$ and $x \to a$ as $h \to 0$, hence

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Considering that $x$ and $a$ are continuous functions, it follows that $x - a$ is continuous so that

$$0 = a - a = \lim_{x \to a} (x - a).$$

Using the fact that the limit of a product is the product of limits (when both limits exist),

$$0 = f'(a) \cdot \lim_{x \to a}(x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}(x - a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot x - a = \lim_{x \to a}[f(x) - f(a)],$$

hence $\lim_{x \to a} f(x) = \lim_{x \to a}[f(a) + f(x) - f(a)] = \lim_{x \to a} f(a) + \lim_{x \to a}[f(x) - f(a)] = f(a)$.

Conversely, the function $|x|$ is continuous on its domain, but it is not differentiable at $a = 0$. $\square$

Computing limits by definition is even more tedious than it looks, but luckily, there are plenty of tools that allow us to compute derivatives of functions without ever touching a limit. Particularly,

- the **Power Rule** says that if $f(x) = x^r$ for some real number $r$, then $f'(x) = rx^{r-1}$;
- the **Product Rule** says that if $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x);$$

- the **Quotient Rule** says that if $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g'(x)]^2}; \text{ and}$$

- the **Chain Rule** says that if $f(x)$ and $g(x)$ are both differentiable, then

$$\frac{d}{dx}[f \circ g(x)] \overset{\text{def}}{=} \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \overset{\text{def}}{=} [f' \circ g(x)] \cdot g'(x).$$

Computing the limit of a function that is continuous is quite easy: we may simply "plug and chug;" however, there exist functions that are not continuous. Even worse, when evaluating limits, we can encounter situations that result in an **indeterminate form** when the limit is the form

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty}.$$

**L'Hôpital's Rule.** Given real functions $f(x)$ and $g(x)$ that are differentiable on an open interval $(a, b)$ except possibly at the point $x = c$ for some real number $a \leq c \leq b$, if

i.) $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \pm\infty$,

ii.) $g'(x) \neq 0$ for any value $a < x < b$ and $x \neq c$, and

iii.) $\lim_{x \to c} \dfrac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

**Example 4.** Compute the limit of $f(x) = \dfrac{\ln x}{x^3 - 1}$ as $x$ approaches $a = 1$.

**Example 5.** Given that $\dfrac{d}{dx} \sin x = \cos x$, compute the limit of $f(x) = \dfrac{\sin x}{x}$ as $x$ approaches $a = 0$.

**Caution:** Unfortunately, this is not a valid proof of this limit identity. In fact, this limit identity is needed to prove that $\frac{d}{dx} \sin x = \cos x$. In order to prove this identity in a rigorous and non-circular manner, we must use tools from trigonometry and the Squeeze Theorem.

**Example 6.** Compute the limit of $f(x) = (2x - \pi) \sec x$ as $x$ approaches $a = \dfrac{\pi}{2}$ from the left.

**Example 7.** Compute the limit of $f(x) = \dfrac{\sin x}{\sin x + \tan x}$ as $x$ approaches $a = 0$.

## Integration and Improper Integrals

Calculus can be divided into two topics — differentiation and integration — that are connected by the Fundamental Theorem of Calculus. Basically, the Fundamental Theorem of Calculus says that differentiation and integration are inverse operations: if $f(x)$ is continuous, then the derivative of the integral of $f(x)$ is $f(x)$, and the integral of $f'(x)$ is $f(x) + C$, where $C$ is a real number.

Consequently, if $f(x)$ measures the velocity of a particle over time, then the (definite) integral of $f(x)$ (over some closed interval) measures the total distance travelled by the particle.

Quite technically, if we denote by $\mathcal{P}$ some collection of points $(x_n, f(x_n))$ on the graph of $f(x)$ with $a = x_1 < x_2 < \cdots < x_n = b$ and $\Delta_i = x_{i+1} - x_i$ for each integer $1 \leq i \leq n-1$, then

$$\int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^{n-1} f(x_i)\,\Delta_i$$

is the **definite integral** of $f(x)$ over the closed interval $[a, b]$, where $\|\mathcal{P}\| = \max\{\Delta_i \mid 1 \leq i \leq n-1\}$. Observe that we may interpret the definite integral $\int_a^b f(x)\,dx$ to be the (signed) area between the curve $f(x)$ and the $x$-axis: the height of a rectangle is $f(x_i)$, and the width is $\Delta_i$. Given that the definite integral of $f(x)$ over the interval $[a, b]$ exists, we say that $f(x)$ is **integrable** on $[a, b]$.

Like we have seen already, the limit definition is often quite cumbersome to use directly, hence we wish to develop some tools that allow us to work with integrals in an optimal way. Our first step toward that is to define an **antiderivative** of $f(x)$ to be a function $F(x)$ such that $F'(x) = f(x)$.

**Example 8.** Prove that the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$.

Observe that for any antiderivative $F(x)$ of a function $f(x)$, there exists a family of antiderivatives indexed by the real numbers. Particularly, the function $G(x) = F(x) + C$ is an antiderivative of $f(x)$ for every real number $C$. Consequently, we may define *the* antiderivative of $f(x)$ to be

$$\int f(x)\,dx = F(x) + C$$

for *any* real number $C$. By the familiar derivative rules, we obtain

- the **Power Rule** for integration, i.e., $\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C$ for all real numbers $r \neq -1$ and
- the **Chain Rule** for integration, i.e., $\int [f' \circ g(x)] \cdot g'(x)\,dx = f \circ g(x) + C$.

Further, we have that $\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$ and $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$ for all real numbers $k$ and all functions $f(x)$ and $g(x)$.

**Example 9.** Compute the antiderivative of $f(x) = \frac{1}{x}$.

**Example 10.** Compute the antiderivative of $f(x) = \sin x \cos x$.

**Example 11.** Compute the antiderivative of $f(x) = xe^{x^2}$.

**The Fundamental Theorem of Calculus, Part I.** Given a function $f(x)$ with an antiderivative $F(x)$ on some closed interval $[a, b]$, the definite integral of $f(x)$ over $[a, b]$ is given by

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

**Example 12.** Compute the (signed) area under the curve $f(x) = x^3$ from $x = 0$ to $x = 1$.

**Example 13.** Compute the (signed) area between $f(x) = \sin x$ and the $x$-axis on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

**The Fundamental Theorem of Calculus, Part II.** Given a function $f(x)$ that is continuous on a closed interval $[a, b]$, for all real numbers $a < x < b$, we have that

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

**Example 14.** Given a differentiable function $g(x)$, use the Fundamental Theorem of Calculus and the Chain Rule for derivatives to prove that

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f'(g(x))g'(x).$$

Our interest in integrals so far has been to find the (signed) area between the curve $f(x)$ and the $x$-axis; however, we have restricted ourselves to finite regions of the $x$-axis. Often, we are interested in how a mathematical model behaves in the long-run, i.e., as $x$ grows arbitrarily large (or approaches $\pm\infty$). Under this framework, we can develop the concept of the improper integral.

Given a function $f(x)$ that is integrable over the closed region $[a, b]$ for every real number $b > a$, the **improper integral** of $f(x)$ over the interval $[a, \infty)$ is defined to be

$$\int_a^\infty f(x)\,dx \stackrel{\text{def}}{=} \lim_{b\to\infty}\int_a^b f(x)\,dx = \lim_{b\to\infty}[F(b) - F(a)]$$

for some antiderivative $F(x)$ of $f(x)$ whenever this limit exists. One can analogously define the improper integral of $f(x)$ over the interval $(-\infty, b]$ whenever $f(x)$ is integrable over the closed region $[a, b]$ for every real number $a < b$ or the doubly improper integral of $f(x)$ over $(-\infty, \infty)$.

**Example 15.** Compute the improper integral $\int_1^\infty x^{-2}\,dx$.

**Example 16.** Compute the improper integral $\int_{-\infty}^1 e^x\,dx$.

**Example 17.** Compute the improper integral $\int_{-\infty}^\infty xe^{-x^2}\,dx$.

Each of the above functions has a horizontal asymptote, hence the improper integrals we computed were all finite. One can also consider the improper integral of a function with a vertical asymptote. Given that $f(x)$ is continuous on the half-open interval $[a, b)$ and $\lim_{x\to b^-} f(x) = \pm\infty$, we have that

$$\int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{t\to b^-}\int_a^t f(x)\,dx = \lim_{t\to b^-}[F(t) - F(a)]$$

for some antiderivative $F(x)$ of $f(x)$ whenever this limit exists. One can analogously define the improper integral of $f(x)$ over the half-open interval $(a, b]$ whenever $\lim_{x\to a^+} f(x) = \pm\infty$.

**Example 18.** Compute the improper integral $\int_0^1 (x-1)^{-1}\,dx$.

**Example 19.** Compute the improper integral $\int_0^1 x^{-1/2}\,dx$.

**Example 20.** Compute the improper integral $\int_{-1}^1 x^{-2/3}\,dx$.

Conventionally, we say that an improper integrals **converges** whenever the limit in question exists, and we say that it **diverges** if the limit does not exist. Even if we cannot explicitly compute an improper integral, the **Comparison Theorem** allows us to say whether it converges or diverges.

**Comparison Theorem for Improper Integrals.** Consider the continuous functions $f(x)$ and $g(x)$ such that $f(x) \geq g(x) \geq 0$ for each real number $x \geq a$.

a.) If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.

b.) If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

One can make analogous statements for the improper integrals $\int_{-\infty}^b f(x)\,dx$ and $\int_{-\infty}^b g(x)\,dx$, doubly improper integrals, and improper integrals of a function with a vertical asymptote.

**Example 21.** Determine the convergence or divergence of the improper integral $\int_0^\infty xe^x\,dx$.

**Example 22.** Determine the convergence or divergence of the improper integral $\int_0^\infty x^{-2}\sin^2 x\,dx$.

## Sequences

One of our main focuses during Calculus II is to understand sequences and — ultimately — series. Unwittingly, we have all encountered sequences in our lives at some point: if you have ever counted while holding your breath, then you have recited a sequence; if you have ever attempted to memorize some of the digits in the decimal expansion of $\pi$, then you have attempted to memorize a sequence; or if you have ever entered a telephone number to place a call, then you have entered into your phone a sequence. Basically, a **sequence** is just an ordered list of numbers. Put more precisely, a sequence is an ordered list $\{a_n\}_{n=1}^k$ of $k$ numbers $a_1, a_2, \dots, a_k$, where $k$ is a positive integer (or whole number). We use the subscript $n$ as an index so that the symbol $a_n$ is the $n$th number that appears in the sequence. Usually, we consider sequences that start with $n = 1$, but it is also possible to think about sequences that begin with any non-negative (or even negative!) whole number index.

Unfortunately, the digits of a telephone number are often quite random, and there is no formula for the $n$th digit in the decimal expansion of $\pi$, so it is impossible to come up with formulae for these sequences; however, there are plenty of sequences for which there exists a formula for the $n$th term. For instance, the natural (or positive whole) numbers $\mathbb{N}$ — obtained by counting up from 1, adding 1 each time — can be listed sequentially as $1, 2, 3, \dots, n, \dots$, hence the infinite sequence

$$\{n\}_{n=1}^\infty = \lim_{k\to\infty} \{n\}_{n=1}^k = \lim_{k\to\infty} \{1, 2, 3, \dots, k\} = \{1, 2, 3, \dots, n, \dots\}$$

consists of all natural numbers. We could also write this sequence as $a_n = n$ for each integer $n \geq 1$. Our interest lies in those sequences (finite or infinite) which we can explicitly write down as $a_n = f(n)$ for all elements $n$ of some index set $N$ that consists of positive whole numbers.

**Example 23.** Find an explicit formula $a_n = f(n)$ for the infinite sequence $-1, 1, -1, 1, \ldots$ that alternates between $-1$ and $1$. Be sure to specify the index set $N$, e.g., $N = \{n \in \mathbb{N} \mid n \geq 1\}$.

**Example 24.** Find an explicit formula $a_n = f(n)$ for the infinite sequence that starts $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, etc. Be sure to specify the index set $N$, e.g., $N = \{n \in \mathbb{N} \mid n \geq 1\}$.

One can define a sequence **recursively** by stating a formula for the $n$th number in the sequence in terms of some of the preceding entries of the sequence. Certainly, the sequence $a_n = n$ for each integer $n \geq 1$ can be written as $a_n = a_{n-1} + 1 = a_{n-2} + 2 = \cdots = a_1 + n - 1$ for each integer $n \geq 1$; however, this is needlessly complicated because we already have a **closed form** for this sequence, i.e., we can already write the sequence $a_n$ as a function $f(n)$ in which $n$ is the only variable.

Curiously enough, even the most simple presentations of a sequence recursively can yield surprisingly complicated closed forms. Consider the **Fibonacci sequence** $a_n = a_{n-1} + a_{n-2}$ for each integer $n \geq 2$ with $a_0 = 0$ and $a_1 = 1$. One can prove that the closed form for this is given by

$$a_n = f(n) = \frac{(-1)^{n-1}\phi^{-n} + \phi^n}{\sqrt{5}}, \text{ where } \phi = \frac{1+\sqrt{5}}{2} \text{ is the Golden Ratio.}$$

**Example 25.** Find a closed form for the sequence $a_n = 2a_{n-1}$ for each integer $n \geq 2$ and $a_1 = 1$.

Computing closed forms for recursive sequences is crucial in the field of computer science, and a rigorous treatment of the subject is often given in any numerical analysis course, but for our purposes, we will not do much more with them than these couple of examples.

Given a sequence $a_n$ for some index set $N$ that consists of positive whole numbers, we can always "reindex" the sequence so that it is defined for each integer $n \geq 1$, so we will assume henceforth that our sequences are all of the form $\{a_n\}_{n=1}^{\infty}$. We say that the sequence $\{a_n\}_{n=1}^{\infty}$ **converges** if there exists some real number $L$ such that for every real number $\varepsilon > 0$, there exists a positive real number $M$ with the property that $|a_n - L| < \varepsilon$ whenever we have that $n > M$. Put another way, the quantity $L$ can be made arbitrarily close to the value of $a_n$ by taking $n$ to be sufficiently large. Given that no such real number $L$ exists, we say that $\{a_n\}_{n=1}^{\infty}$ **diverges**. Further, if the terms of $a_n$ increase (or decrease) without bound, then $a_n$ **diverges to infinity** (or negative infinity).

Our prototypical example of a convergent sequence is the sequence of reciprocals of natural numbers $a_n = \frac{1}{n}$. Observe that as $n$ grows, the reciprocals $\frac{1}{n}$ become smaller but remain positive. Consequently, we suspect that $\lim_{n \to \infty} a_n = 0$. Let us prove this. Given any real number $\varepsilon > 0$, we want to find a positive real number $M$ such that whenever $n > M$, we have that $\left|\frac{1}{n}\right| < \varepsilon$. Considering that $n \geq 1$, we have that $\frac{1}{n} > 0$ so that $\left|\frac{1}{n}\right| = \frac{1}{n}$. We can ensure that $\frac{1}{n} < \varepsilon$ by taking $n > \frac{1}{\varepsilon}$, hence our choice for $M$ is obvious: we should take $M$ to be $\frac{1}{\varepsilon}$. Unwinding this gives a formal proof.

*Proof.* We claim that $\lim_{n \to \infty} \frac{1}{n} = 0$. Given any real number $\varepsilon > 0$, if we have that $n > M = \frac{1}{\varepsilon}$, then

$$\left|\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{M} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon.$$ □

**Example 26.** Compute the limit of the sequence $a_n = \dfrac{1}{2^n}$, or prove that it does not exist.

**Example 27.** Compute the limit of the sequence $a_n = (-1)^n$, or prove that it does not exist.

By definition, the limit of an infinite sequence $a_n$ depends only on the values that $a_n$ takes for sufficiently large indices $n$. Given some arbitrarily large (but fixed) positive real number $M$, we refer to the values of $a_n$ for all indices $n > M$ as the **$M$-tail** of the sequence $a_n$. Consequently, the limit of an infinite sequence depends only on the $M$-tail of $a_n$, and as such, it will not be altered if we change (or omit) finitely many terms — namely, all of those terms $a_n$ for $n \leq M$. Further, if there exists a real number $C$ such that $a_n = C$ for all indices $n > M$, then $\lim_{n\to\infty} a_n = C$.

Other than the Fibonacci sequence, we have studied (and will primarily study) only sequences $\{a_n\}_{n=1}^{\infty}$ with a closed form, i.e., infinite sequences for which there exists a function $f(n)$ such that $a_n = f(n)$ for each integer $n \geq 1$. Consequently, we can think about sequences as functions whose domains have been restricted to the positive whole numbers. Using the tools that we have from Calculus I — limits, derivatives, L'Hôpital's Rule, etc. — we can better understand sequences with closed forms in terms of the functions that define them. Particularly, we have the following fact.

**Fact:** Given a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n = f(n)$ for some function $f : \mathbb{R}_{>0} \to \mathbb{R}$, we have that

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$

*Proof.* Compare the definitions to see that this is true. Explicitly, if there exists a real number $L$ such that $\lim_{x\to\infty} f(x) = L$, then by definition, given a real number $\varepsilon > 0$, there exists a positive real number $M$ such that $|f(x) - L| < \varepsilon$ for all real numbers $x > M$. But if this is true for all real numbers $x > M$, then it is certainly true for all positive whole numbers $n > M$ so that $\lim_{n\to\infty} a_n = L$. Use the analogous argument in the case that $\lim_{x\to\infty} f(x) = \pm\infty$ to show that $\lim_{n\to\infty} a_n = \pm\infty$. □

**Example 28.** Compute the limit of the sequence $a_n = \dfrac{\sin n}{n}$.

**Example 29.** Compute the limit of the sequence $a_n = \dfrac{\ln n}{n}$.

**Example 30.** Compute the limit of the sequence $a_n = \dfrac{n^4 - 5n^3 + 3n^2 + 1}{3n^4 - 7n^2 + n + 1}$.

Given any real numbers $c \neq 0$ and $r \geq 0$, we refer to a sequence of the form $a_n = cr^n$ as a **geometric sequence**. Considering that $a_n$ measures the hypervolume of the $n$-dimensional hypercube of side length $r$ that has been dilated by a factor of $c$, the sequence $a_n = cr^n$ does indeed describe something geometric — hence the name. We refer to the constant $r$ as the **common ratio** of the geometric series since it can be obtained by taking the ratio of each term with its preceding term:

$$r = r^{n+1-n} = \frac{r^{n+1}}{r^n} = \frac{cr^{n+1}}{cr^n} = \frac{a_{n+1}}{a_n} \text{ for each integer } n \geq 1.$$

We can completely classify the convergence of geometric sequences by the following fact.

**Fact:** Given any real number $c \neq 0$, the geometric sequence $a_n = cr^n$ satisfies

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} cr^n = \begin{cases} 0 & \text{if } 0 \leq r < 1; \\ c & \text{if } r = 1; \text{ and} \\ \infty & \text{if } r > 1. \end{cases}$$

*Proof.* Consider the function $f(x) = cr^x$. Observe that $a_n = f(n)$ for each integer $n \geq 1$, hence

$$\lim_{n\to\infty} cr^n = \lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} cr^x = c \cdot \lim_{x\to\infty} r^x.$$

Certainly, this limit is 0 if $r = 0$. Consequently, we will assume that $r > 0$ so that $r^x > 0$. We have therefore that $r^x = e^{\ln r^x} = e^{x \ln r}$. Given that $r < 1$, it follows that $\ln r < 0$ so that

$$\lim_{n\to\infty} cr^n = c \cdot \lim_{x\to\infty} r^x = c \cdot \lim_{x\to\infty} e^{x\ln r} = \lim_{t\to-\infty} e^t = 0,$$

where we have used the fact that if $t = x \ln r$, then $t \to -\infty$ as $x \to \infty$. Given that $r = 1$, it follows that $\ln r = 0$ so that $r^x = e^{x \ln r} = e^0 = 1$, from which it follows that

$$\lim_{n\to\infty} cr^n = c \cdot \lim_{x\to\infty} r^x = c \cdot \lim_{x\to\infty} 1 = c.$$

Given that $r > 1$, it follows that $\ln r > 0$ so that

$$\lim_{n\to\infty} cr^n = c \cdot \lim_{x\to\infty} r^x = c \cdot \lim_{x\to\infty} e^{x\ln r} = \lim_{t\to\infty} e^t = \infty. \qquad \square$$

**Example 31.** Determine if the sequence $a_n = \dfrac{1}{3^{3n-2}}$ is geometric. If so, find the constant $c$ and the common ratio $r$, and determine with justification if $a_n$ converges or diverges; if not, explain why.

**Example 32.** Determine if the sequence $6, -3, \frac{3}{2}, -\frac{3}{4}$, etc. is geometric. If so, find the constant $c$ and the common ratio $r$, and determine with justification its convergence; if not, explain why.

**Example 33.** Determine if the sequence $a_n = \ln e^\pi$ is geometric. If so, find the constant $c$ and the common ratio $r$, and determine with justification if $a_n$ converges or diverges; if not, explain why.

Considering that the definition of the limit of a sequence is closely related to the definition of the limit of a function at $\pm\infty$, it is not surprising that the familiar limit laws holds for sequences.

**Limit Laws for Sequences.** Given convergent sequences with $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$,

i.) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M$;

ii.) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM$; and

iii.) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \dfrac{L}{M}$ whenever $M \neq 0$.

Likewise, there is an analog of the Squeeze Theorem for sequences.

**Squeeze Theorem for Sequences.** Given any sequences $a_n$, $b_n$, and $c_n$ such that

i.) $b_n \leq a_n \leq c_n$ for all integers $n > M$ for some positive real number $M$ and

ii.) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$,

we have that $\lim_{n\to\infty} a_n = L$.

**Example 34.** Prove that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

**Example 35.** Given any real number $c \neq 0$, prove that the geometric sequence $a_n = cr^n$ satisfies

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} cr^n = \begin{cases} 0 & \text{if } 1 < r < 0 \text{ and} \\ \text{DNE} & \text{if } r \leq -1. \end{cases}$$

**Example 36.** Given any real number $c$, prove that $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

**Hint:** Of course, if $c = 0$, then this is clear because the sequence is constantly 0. Consider the case that $M \leq c < M + 1$ for some integer $M \geq 0$. Consider the $n$th term of the sequence.

$$\frac{c^n}{n!} = \frac{c \cdot c \cdot c \cdots c \cdot c \cdot c \cdots c}{1 \cdot 2 \cdots M \cdot (M+1) \cdot (M+2) \cdots (n-1) \cdot n} = \underbrace{\frac{c}{1} \cdot \frac{c}{2} \cdot \frac{c}{3} \cdots \frac{c}{M}}_{\text{Call this constant } R.} \cdot \underbrace{\frac{c}{M+1} \cdot \frac{c}{M+2} \cdots \frac{c}{n-1}}_{\text{Each term here is } <1.} \cdot \frac{c}{n}$$

Use the Squeeze Theorem and Example 34 to finish the proof.

Continuous functions are characterized by the property that $\lim_{x\to L} f(x) = f(L)$, i.e., the limit can be pushed inside the function. Luckily, the same holds for limits of continuous functions of sequences.

**Limits Commute with Continuous Functions.** Given a continuous function $f(x)$ and a convergent sequence $a_n$ such that $\lim_{n\to\infty} a_n = L$, the sequence $f(a_n)$ is convergent, and its limit is

$$\lim_{n\to\infty} f(a_n) = f\big(\lim_{n\to\infty} a_n\big) = f(L).$$

**Example 37.** Compute the limit of the sequence $a_n = \sin(e^{-n})$.

We have already fully classified the convergence of geometric sequences. Consequently, one naturally questions whether we can fully classify the convergence of all sequences.

Our first step toward that goal is to study sequences that are bounded. We say that a sequence $a_n$ is **bounded above** if there exists a real number $M^+$ such that $a_n \leq M^+$ for all integers $n \geq 1$. Likewise, we say that a sequence is **bounded below** if there exists a real number $M^-$ such that $a_n \geq M^-$ for all integer $n \geq 1$. Combining these two notions, we say that a sequence is **bounded** whenever it is bounded above and bounded below. Given that this is not the case (i.e., the sequence is either not bounded above or not bounded below), we say that the sequence is **unbounded**.

**Convergent Sequences Are Bounded.** If the sequence $a_n$ converges, then $a_n$ is bounded.

*Proof.* Given that $a_n$ converges, there exists a real number $L$ such that $\lim_{n\to\infty} a_n = L$. By definition of the limit, there exists a positive real number $N$ such that $|a_n - L| < 1$ for all $n > N$. Unraveling this gives that $L - 1 < a_n < L + 1$ for all $n > N$. We can choose $M^+$ to be larger than $a_1, a_2, \ldots, a_N$ and $L + 1$. Likewise, we can choose $M^-$ to be smaller than $a_1, a_2, \ldots, a_N$ and $L - 1$. $\square$

Essentially, the proof of the above fact uses the definition of the limit of $a_n$ to explicitly construct an upper and lower bound for $a_n$. We refer to such a proof as a *constructive* proof.

Contrapositively, this fact states that if $a_n$ is unbounded, then $a_n$ diverges. On the other hand, we have already encountered bounded but divergent sequences, e.g., $a_n = (-1)^n$. Quite generally, every oscillating sequence is bounded and divergent. We say that a sequence is **oscillating** if there exist (at least) two distinct constants at which the sequence takes values for infinitely many indices.

Given that a sequence $a_n$ is (eventually) monotone, then its boundedness *is* sufficient to conclude convergence. We say that $a_n$ is **monotone** if it is either increasing or decreasing. Particularly, $a_n$ is increasing (or decreasing) if $a_n \leq a_{n+1}$ (or $a_n \geq a_{n+1}$) for all $n > M$ for some constant $M$.

**Criterion for Monotonicity.** Given a sequence $a_n = f(n)$ for some function $f : \mathbb{R}_{>0} \to \mathbb{R}$, if $f(x)$ is differentiable and $f'(x) \geq 0$ (or $f'(x) \leq 0$) for all $x > M$, then $a_n$ is increasing (or decreasing).

**Example 38.** Determine whether the sequence $a_n = \sin\left(\frac{1}{n}\right)$ is monotone.

**Example 39.** Determine whether the sequence $a_n = -ne^{-n^2}$ is monotone.

**Example 40.** Determine whether the sequence $a_n = \cos(\pi n)$ is monotone.

**Monotone Convergence Theorem.** A monotone sequence $a_n$ converges if and only it is bounded. Explicitly, if $a_n$ is increasing and bounded above by $M^+$, then $a_n$ converges and $\lim_{n\to\infty} a_n \leq M^+$. Likewise, if $a_n$ is decreasing and bounded below by $M^-$, then $a_n$ converges and $\lim_{n\to\infty} a_n \geq M^-$.

**Example 41.** Determine whether the sequence $a_n = \sin\left(\frac{1}{n}\right)$ converges. If so, find the limit.

**Example 42.** Determine whether the sequence $a_n = -ne^{-n^2}$ converges. If so, find the limit.

**Example 43.** Determine whether the sequence $a_n = \cos(\pi n)$ converges. If so, find the limit.

# Basics of Infinite Series

One of the most powerful and important tools in all of mathematics is the infinite series. Countless uses for series abound in approximation theory, real analysis, complex analysis, combinatorics, probability, and statistics. Concretely, infinite series can be used to approximate $\pi$ (and many other irrational numbers) to any desired degree of accuracy: we will eventually learn that

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \cdots .$$

Of course, we have already familiarized ourselves with finite series: the Riemann sum

$$\sum_{k=1}^{n-1} f(x_k)\,\Delta x_k = f(x_1)\,\Delta x_1 + \cdots + f(x_{n-1})\,\Delta x_{n-1}$$

for some function $f(x)$ and some sequence of points $\{x_k\}_{k=1}^{n-1}$ with $\Delta x_k = x_{k+1} - x_k$ for each integer $1 \leq k \leq n-1$ is a finite series from Calculus I. We refer to this presentation of the sum as sigma notation (named for the Greek letter sigma $\Sigma$). Certainly, a finite series can be evaluated by simply adding up all of its terms, hence we are interested in evaluating infinite series (when possible).

We define an infinite series as the limit of the partial sums of a finite series, i.e.,

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n) \overset{\text{def}}{=} \lim_{n\to\infty} s_n,$$

where $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$ is the $n$**th partial sum** of the infinite series. Consequently, we can think of an infinite series as the limit of the sequence $\{s_n\}_{n=1}^{\infty}$ of its partial sums.

**Example 44.** Find the first five partial sums of the infinite series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$.

Considering an infinite series as the limit of its sequence of partial sums, we can apply all of the techniques from section 10.1 to our study of infinite series. Particularly, we have the following fact.

**Convergence of Partial Sums Implies Convergence of Infinite Series.** Given any whole number $m \geq 1$, the infinite series $\sum_{k=m}^{\infty} a_k$ converges if and only if the sequence $s_n = \sum_{k=m}^{n} a_k$ converges. Further, we have that $\sum_{k=m}^{\infty} a_k = s = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \sum_{k=m}^{n} a_k$. Given that $s$ does not exist, we say that the series **diverges**. Given that $s = \pm\infty$, we say that the series **diverges to infinity**.

**Example 45.** Find an explicit formula for the partial sums of the infinite series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k}$; then, determine whether the infinite series converges. If so, find its value.

**Hint:** Use the fact that $1 + 2 + \cdots + 2^{n-1} = 2^n - 1$ to find $s_n$.

Certain types of infinite series are easier to compute than others. One of these is the **telescoping** series whose $n$th partial sum can be written as $s_n = c + f(n)$ for some constant $c$ and some function $f : \mathbb{R}_{>0} \to \mathbb{R}$. Consequently, the telescoping series converges if and only if $s_n = c + f(n)$ converges if and only if $f(n)$ converges, and its value is equal to $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} [c + f(n)] = c + \lim\limits_{n\to\infty} f(n)$.

**Example 46.** Determine if the infinite series $\displaystyle\sum_{k=1}^{\infty} \left( \frac{1}{2k} - \frac{1}{2k+2} \right)$ converges. If so, find its value.

Using the technique of partial fraction decomposition, one can recognize many infinite series as telescoping series, rendering them far easier to compute than meets the eye.

Given a geometric sequence $a_n = cr^n$ for some nonzero real numbers $c$ and $r$, the infinite series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} cr^n$ is said to be a **geometric series**. Like with geometric sequences, we can completely classify convergence of a geometric series based on the common ratio $r$.

**Fact:** Given any real number $c \neq 0$, the geometric series $\sum_{n=0}^{\infty} cr^n$ satisfies

$$\sum_{n=0}^{\infty} cr^n = \begin{cases} \dfrac{c}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

*Proof.* Given that $r = 1$, the $n$th partial sum of the geometric series is given by

$$s_n = \underbrace{c + c + \cdots + c}_{n \text{ summands}} = n \cdot c.$$

Consequently, we have that $\lim_{n \to \infty} n \cdot c = c \cdot \lim_{n \to \infty} n = \infty$, and the geometric series diverges to $\infty$.

Given that $r \neq 1$, observe that $(1 + r + \cdots + r^n)(1 - r) = 1 - r^{n+1}$, from which it follows that

$$s_n = c(1 + r + \cdots + r^n) = c \cdot \frac{1 - r^{n+1}}{1 - r}$$

is the $n$th partial sum of the geometric series. Consequently, we have that

$$\lim_{n \to \infty} s_n = c \cdot \lim_{n \to \infty} \frac{1 - r^{n+1}}{1 - r} = c \cdot \frac{1 - \lim_{n \to \infty} r^{n+1}}{1 - r},$$

and this diverges whenever $|r| \geq 1$ and $r \neq 1$ and converges to $\dfrac{c}{1-r}$ whenever $|r| < 1$. $\square$

**Fact:** Given any real numbers $c$ and $r$ with $|r| < 1$, we have that $\displaystyle\sum_{n=k}^{\infty} cr^n = \frac{cr^k}{1-r}$.

*Proof.* By the fact, above, we have that

$$\sum_{n=k}^{\infty} cr^n = \sum_{n=0}^{\infty} cr^n - (c + cr + \cdots + cr^{k-1}) = \frac{c}{1-r} - \frac{c(1 - r^k)}{1-r} = \frac{cr^k}{1-r}.$$

$\square$

**Example 47.** Determine if the infinite series $\displaystyle\sum_{n=1}^{\infty} (\ln e^{\pi})^n$ converges. If so, find its value.

Later in the course, we will consider infinite series as the discrete analog of improper integrals. Like with convergent integrals, there are nice linearity properties for convergent series.

**Linearity of Convergent Series.** Given convergent series $\sum a_n$ and $\sum b_n$, we have that

(i.) $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$ and

(ii.) $\sum ca_n = c \cdot \sum a_n$ for any constant $c$.

Particularly, any linear combination of convergent series is a convergent series.

We have thus far determined when telescoping and geometric series are convergent. Conversely, we can determine when an infinite series is divergent by inspecting its summands.

**The Divergence Test.** Given that $\lim_{n\to\infty} a_n \neq 0$, we have that $\sum a_n$ is divergent.

*Proof.* Observe that the partial sums of $\sum a_n$ are given by $s_n = a_1 + \cdots + a_{n-1} + a_n$ and $s_{n-1} = a_1 + \cdots + a_{n-1}$ so that $s_n = s_{n-1} + a_n$ and $a_n = s_n - s_{n-1}$. Given that $\sum a_n$ converges, it follows that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_{n-1} = s$ so that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - s_{n-1} = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$ $\square$

On first glance, it might appear that the above proof did not actually establish what we intended: we proved that if $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. We refer to this is a *proof by contrapositive.*

**Example 48.** Determine if the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$ converges. If so, find its value.

**Example 49.** Determine if the infinite series $\displaystyle\sum_{n=7}^{\infty} \frac{n^3+n^2+n+1}{n^3-n^2+n-1}$ converges. If so, find its value.

**Caution.** Often, upon first learning the Divergence Theorem, students get mixed up in the logic of what exactly the theorem guarantees. Put explicitly, the theorem says that

1.) if the limit of the sequence $a_n$ of terms of the series *does not converge to* 0, then it is impossible for the series $\sum a_n$ to converge, and

2.) if the series $\sum a_n$ converges, then the sequence $a_n$ of terms of the series *must converge to* 0.

Consequently, we are able to decipher when a series *diverges* by the Divergence Test — hence the name; however, the drawback is that we cannot tell that a series *converges* by the Divergence Test.

**The Converse of the Divergence Test Is False.** There exists a sequence $a_n$ with $\lim_{n\to\infty} a_n = 0$ such that the infinite series $\sum a_n$ diverges.

*Proof.* Consider the sequence $a_n = \frac{1}{n}$. We have seen in the exposition preceding Example 26 that $\lim_{n\to\infty} a_n = 0$. On the other hand, observe that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\ &> \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots \\ &= \frac{1}{1} + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \cdots \\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\ &= \infty. \end{aligned}$$

$\square$

Considering that this is our prototypical counterexample to the converse of the Divergence Test, we give it a name: we refer to the divergent infinite series $\sum \frac{1}{n}$ as the *harmonic series.*

**Example 50.** Prove that $a_n = \dfrac{1}{\sqrt{n}}$ satisfies $\lim\limits_{n \to \infty} a_n = 0$; then, prove that $\sum a_n$ diverges.

**Hint:** Toward a proof of the second statement, we may first find a suitable lower bound for the sequence $s_n$ of partial sums of $\sum a_n$. We claim then that $\lim\limits_{n \to \infty} s_n = \infty$.

# Convergence Tests for Series

Given a sequence $a_k$ such that $a_k \geq 0$ for all $k$ sufficiently large, one immediate interpretation of the infinite series $\sum_{k=m}^\infty a_k = a_m + a_{m+1} + \cdots$ is as the area of the (infinitely many) rectangles with width 1 and height $a_k$ for each integer $k \geq m$. Consequently, it follows that the sequence $s_n$ of partial sums of $\sum_{k=m}^\infty a_k$ is an increasing sequence, and we can apply the Monotone Convergence Theorem to determine the convergence of the infinite series by the fact that $\sum_{k=m}^\infty a_k = \lim\limits_{n \to \infty} \sum_{k=m}^n a_k = \lim\limits_{n \to \infty} s_n$.

**Convergence of a Series with Positive Terms.** Given a sequence $a_k$ such that $a_k \geq 0$ for all integers $k$ sufficiently large,

(a.) if the sequence $s_n$ of partial sums is bounded above, then $\sum_{k=m}^\infty a_k$ converges, and

(b.) if the sequence $s_n$ of partial sums is not bounded above, then $\sum_{k=m}^\infty a_k$ diverges.

Often, in practice, it is difficult to find a closed form for the partial sums of a series, hence it is difficult to determine if the sequence of partial sums is bounded above. Luckily, we can say more.

**The Integral Test.** Given a sequence $a_k = f(k)$ for some function $f(x)$ that is positive, decreasing, and continuous for all real numbers $x \geq m$, then

(i.) if $\int_m^\infty f(x)\,dx$ converges, then $\sum_{n=m}^\infty a_n$ converges, and

(ii.) if $\int_m^\infty f(x)\,dx$ diverges, then $\sum_{n=m}^\infty a_n$ diverges.

*Proof.* By our above interpretation, the infinite series $\sum_{k=m}^\infty a_k$ represents the area of the (infinitely many) rectangles with width 1 and height $a_n = f(n)$. Considering that $f(x)$ is decreasing, it follows that the right endpoint approximation is an underestimate, i.e., $a_{m+1} + \cdots + a_b \leq \int_m^b f(x)\,dx$ for all integers $b \geq m+1$. We conclude therefore that $\sum_{n=m}^\infty a_n = \lim\limits_{b \to \infty} \sum_{n=m}^b a_n \leq \lim\limits_{b \to \infty} \int_m^b f(x)\,dx$. On the other hand, the left endpoint approximation is an overestimate, i.e., $a_m + \cdots + a_{b-1} \geq \int_m^b f(x)\,dx$ for all integers $b \geq m+1$. Consequently, we have that $\sum_{n=m}^\infty a_n = \lim\limits_{b \to \infty} \sum_{n=m}^b a_n \geq \lim\limits_{b \to \infty} \int_m^b f(x)\,dx$. $\square$

**Example 51.** Use the Integral Test to prove that $\displaystyle\sum_{n=m}^\infty \frac{1}{n}$ diverges for any positive integer $m$.

**Example 52.** Use the Integral Test to determine the convergence of $\displaystyle\sum_{n=m}^\infty \frac{1}{1+n^2}$.

Using facts about $\int_m^\infty \frac{1}{x^p}$ for some real number $p$ and positive integer $m$, we can use the Integral Test to determine the convergence (or divergence) of a family of series known as *p-series.*

**The $p$-Series Test.** Given a real number $p \neq 0$, we refer to $\displaystyle\sum_{n=m}^\infty \frac{1}{n^p}$ as a *p-series.*

(i.) If $p > 1$, then the $p$-series $\displaystyle\sum_{n=m}^\infty \frac{1}{n^p}$ converges.

(ii.) If $p \leq 1$, then the $p$-series $\displaystyle\sum_{n=m}^\infty \frac{1}{n^p}$ diverges.

*Proof.* Given that $p \leq 0$, we have that $\frac{1}{n^p} \geq 1$ for all $n \geq m$, hence the series diverges by the Divergence Test. Given that $p > 0$, the function $\frac{1}{x^p}$ is positive, decreasing, and continuous for all $x \geq m$. By the Integral Test, we have that $\sum_{n=m}^\infty \frac{1}{n^p}$ converges if and only if $\int_m^\infty \frac{1}{x^p}\,dx$ converges. Evaluating this improper integral in either case establishes the result. Explicitly, we have that

$$\int_m^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_m^b x^{-p}\,dx = \begin{cases} \displaystyle\lim_{b\to\infty} \frac{x^{1-p}}{1-p}\bigg|_m^b & \text{if } p \neq 1 \text{ and} \\ \\ \displaystyle\lim_{b\to\infty} \ln x\bigg|_m^b & \text{if } p = 1. \end{cases}$$

Given that $p > 1$, it follows that $1 - p < 0$ so that the limit converges. On the other hand, if $p < 1$, then $1 - p > 0$ so the limit diverges to infinity. Certainly, the limit of $\ln x$ diverges to infinity. $\square$

**Example 53.** Use the $p$-Series Test to determine the convergence of $\displaystyle\sum_{n=m}^\infty \frac{1}{\sqrt[5]{n^7}}$.

**Example 54.** Use the $p$-Series Test to determine the convergence of $\displaystyle\sum_{n=m}^\infty \frac{1}{\sqrt[7]{n^5}}$.

Of course, the $p$-Series Test only applies to the reciprocal of power functions, so it is not (immediately) applicable to determine the convergence of series the likes of $\sum_{n=0}^\infty \frac{n^2}{1+n^5}$. Further, we would not want to endeavor to use the Integral Test on such a series because the antiderivative of the function $\frac{x^2}{1+x^5}$ is absolutely horrendous. But rest assured, we are not out of luck!

**The Direct Comparison Test.** Given sequences $a_n$ and $b_n$ such that there exists a real number $M$ with $0 \leq a_n \leq b_n$ for all integers $n > M$,

(i.) if $\sum b_n$ converges, then $\sum a_n$ converges and

(ii.) if $\sum a_n$ diverges, then $\sum b_n$ diverges.

*Proof.* By hypothesis that $0 \leq a_n \leq b_n$ for all integers $n > M$, we have that

$$a_n + \cdots + a_N \leq b_n + \cdots + b_N$$

for all integers $N \geq n > M$. Consequently, if $\sum b_n$ converges, then the sequence $s_n$ of partial sums of $\sum b_n$ is bounded above by the Monotone Convergence Theorem since $s_n$ is increasing and convergent. We have therefore that the sequence $t_n$ of partial sums of $\sum a_n$ is bounded above, and the Monotone Convergence Theorem guarantees that $\sum a_n$ converges because $t_n$ is increasing and bounded above. Considering that (ii.) is the contrapositive to (i.), our proof is complete. $\square$

**Example 55.** Use the Direct Comparison Test to determine the convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{1+n^5}$.

**Example 56.** Use the Direct Comparison Test to determine the convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt[7]{n^5+1}}$.

**Hint:** Establish that there exists a real number $M$ with $0 \leq \dfrac{1}{n} \leq \dfrac{1}{\sqrt[7]{n^5+1}}$ for all integers $n > M$. Observe that this is equivalent to showing that $n \geq \sqrt[7]{n^5+1}$ or $n^7 - n^5 - 1 \geq 0$.

Our previous example shows that the Direct Comparison Test can be employed to test for convergence of the reciprocal of a composition of a polynomial and a power function; however, this application is not always the most straightforward, as we have seen in Example 56: our knee-jerk reaction is to compare to the divergent $p$-series with $p = 5/7$; however, this does not result in anything useful. Once again, there is a more powerful test that we can use instead.

**The Limit Comparison Test.** Given sequences $a_n, b_n \geq 0$, consider the limit $L = \displaystyle\lim_{n \to \infty} \frac{a_n}{b_n}$.

(i.) If $L > 0$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

(ii.) If $L = \infty$, then if $\sum a_n$ converges, then $\sum b_n$ converges.

(iii.) If $L = 0$, then if $\sum b_n$ converges, then $\sum a_n$ converges.

*Proof.* We will assume first that $L \geq 0$. Consequently, we may find a real number $R > L$ such that $0 \leq \frac{a_n}{b_n} \leq R$ for all integers $n$ sufficiently large. We have therefore that $0 \leq a_n \leq Rb_n$. By the Direct Comparison Test, if $\sum b_n$ converges, then $\sum a_n$ converges, hence the forward direction of part (i.) and part (iii.) are both established. We will assume now that $L > 0$ or $L = \infty$. Consider the limit

$$K = \lim_{n \to \infty} \frac{b_n}{a_n}.$$

Observe that $K = \frac{1}{L}$ is finite. Particularly, we have that $K \geq 0$. By the previous paragraph, we may reverse the roles of $a_n$ and $b_n$ to conclude the converse of part (i.) and part (ii.). $\square$

Unfortunately, though this proof is quite clever, it obscures the intuition behind the Limit Comparison Test. Essentially, one can rephrase the three statements in the following manner.

(i.) If $L > 0$, then for $n$ sufficiently large, we have that $a_n \approx Lb_n$. Consequently, $\sum a_n$ converges if and only if $\sum b_n$ converges because these series are scalar multiples of each other.

(ii.) If $L = \infty$, then the sequence $a_n$ is eventually significantly larger than $b_n$, hence if $\sum a_n$ converges, then $\sum b_n$ must also converge (by the Direct Comparison Test).

(iii.) If $L = 0$, then the sequence $b_n$ is eventually significantly larger than $a_n$, hence if $\sum b_n$ converges, then $\sum a_n$ must also converge (by the Direct Comparison Test).

**Example 57.** Use the Limit Comparison Test to determine the convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt[7]{n^5+1}}$.

**Example 58.** Use the Limit Comparison Test to determine the convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n^3 - n^2 + n - 1}{n^4 - n^3 + n^2 - n + 1}$.

# Absolute and Conditional Convergence

Our study of infinite series so far has given us the tools to recognize many different families of series of which we can readily determine the convergence. Geometric series are of the form $\sum cr^n$ for some real numbers $c$ and $r$, and these converge if and only if $|r| < 1$. We have also discussed the family of $p$-series $\sum \frac{1}{n^p}$ for some real number $p$: these converge if and only if $p > 1$.

Last section, we developed a broad array of tests to determine convergence of series with positive terms (or only finitely many negative terms). Unfortunately, we have yet to concern ourselves with non-geometric series that contain infinitely many negative terms. For instance, we know that the harmonic series $\sum \frac{1}{n}$ diverges (as it is a $p$-series with $p = 1$); however, we shall soon discover the curious fact that the *alternating harmonic series* $\sum (-1)^n \frac{1}{n}$ converges!

Considering that we have many ways to determine if a series of positive terms converges, it makes sense to think about the series $\sum |a_n|$ for some sequence $a_n$. We say that the series $\sum a_n$ **converges absolutely** (or $\sum a_n$ is **absolutely convergent**) whenever the series $\sum |a_n|$ converges.

**Example 59.** Determine if the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^{-\pi}$ is absolutely convergent.

**Absolute Convergence Implies Convergence.** If $\sum |a_n|$ converges, then $\sum a_n$ converges.

*Proof.* By definition of the absolute value, we have that $-|a_n| \leq a_n \leq |a_n|$ from which it follows that $0 \leq a_n + |a_n| \leq 2|a_n|$. By the Direct Comparison Test, if $\sum |a_n|$ converges, then $\sum (a_n + |a_n|)$ converges. Further, we have that $\sum a_n = \sum (a_n + |a_n| - |a_n|) = \sum (a_n + |a_n|) - \sum |a_n|$ converges. $\square$

**Example 60.** Determine if the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^{-2}$ converges.

Like we mentioned above, the alternating harmonic series $\sum (-1)^n \frac{1}{n}$ converges; however, the harmonic series $\sum \frac{1}{n}$ does not converge. Generally, we say that the series $\sum a_n$ **converges conditionally** (or $\sum a_n$ is **conditionally convergent**) whenever $\sum a_n$ converges and the series $\sum |a_n|$ diverges. Consequently, alternating harmonic series is conditionally convergent. But why?

**The Alternating Series Test.** Given a series $\sum a_n$ such that $a_n = (-1)^n b_n$ for some <u>positive</u> and <u>decreasing</u> sequence $b_n$ such that $\lim_{n \to \infty} b_n = 0$, the series $\sum a_n$ converges.

**Example 61.** Prove that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges.

**Example 62.** Determine all values of $p$ such that the alternating $p$-series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$ converges. Explain how this differs from the case of the non-alternating $p$-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$.

# The Ratio Test

We have presented thus far many tests for determining the convergence of infinite series; however, we have always paid strong attention to the sign of the terms of the series. Particularly, we cannot apply the Integral Test or either of the Comparison Tests to a series whose terms alternate in sign. On the other hand, we cannot apply the Alternating Series Test to a series with positive terms. Our last two series tests can be applied to any series regardless of the sign of the general terms.

**The Ratio Test.** Given the series $\sum a_n$, consider the limit

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

(a.) If $L < 1$, then $\sum a_n$ converges absolutely.

(b.) If $L > 1$, then $\sum a_n$ diverges.

(c.) If $L = 1$, then the series could converge or diverge.

*Proof.* We can easily dispense of the case that $L = 1$: the infinite series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ both satisfy $L = 1$, but the former diverges, and the latter converges by the $p$-Series Test.

By definition of the limit, given any real number $\varepsilon > 0$, there exists a real number $M$ such that for all integers $n \geq M$, we have that $-\varepsilon < |a_{n+1}/a_n| - L < \varepsilon$. By simplifying this expression, we have that $L - \varepsilon < |a_{n+1}/a_n| < L + \varepsilon = r$ for all integers $n \geq M$. Given that $L < 1$, we can ensure that $r < 1$ by taking $\varepsilon$ to be sufficiently small. Observe that we have

$$\begin{aligned} &|a_{M+1}| < |a_M| r, \\ &|a_{M+2}| < |a_{M+1}| r < |a_M| r^2, \\ &|a_{M+3}| < |a_{M+2}| r < |a_{M+1}| r^2 < |a_M| r^3, \end{aligned}$$

and in general, $|a_{M+n}| < |a_M| r^n$. Consequently, we have that

$$\sum_{n=M}^{\infty} |a_n| = \sum_{k=0}^{\infty} |a_{M+k}| = \sum_{n=0}^{\infty} |a_{M+n}| < \sum_{n=0}^{\infty} |a_M| r^n = |a_M| \sum_{n=0}^{\infty} r^n.$$

By the Geometric Series Test, the geometric series on the right-hand side converges by hypothesis that $0 \leq L < r < 1$. By the Direct Comparison Test, the series $\sum |a_n|$ converges, so $\sum a_n$ converges absolutely by definition. We need only prove the remaining case (b.).

By the previous paragraph, given any real number $\varepsilon > 0$, there exists a real number $M$ such that for all integers $n \geq M$, we have that $|a_{n+1}/a_n| > L - \varepsilon = r$. Given that $L > 1$, we can ensure that $r > 1$ by taking $\varepsilon$ to be sufficiently small. By a similar argument as before, it follows that $|a_{M+n}| > |a_M| r^n$. Considering that $r > 1$, it follows that

$$\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} |a_{M+n}| > \lim_{n \to \infty} |a_M| r^n = |a_M| \lim_{n \to \infty} r^n = \infty.$$

Consequently, we have that $\lim_{n \to \infty} a_n \neq 0$, hence $\sum a_n$ diverges by the Divergence Test. □

**Example 63.** Use the Ratio Test to determine if the series $\sum_{n=0}^{\infty} \frac{e^n}{n!}$ converges.

**Example 64.** Use the Ratio Test to determine if the series $\sum_{n=1}^{\infty} \frac{n^n}{(n^2)!}$ converges.

# Power Series and Taylor Series

Recall from precalculus that a polynomial of degree $n$ is a function of the form $a_0 + a_1 x + \cdots + a_n x^n$ for some real numbers $a_0, a_1, \ldots, a_n$ such that $a_n$ is nonzero and $a_i$ is the coefficient of the monomial $x^i$ of degree $i$ for each integer $0 \leq i \leq n$. We refer to the monomials $a_i x^i$ as *terms* of the polynomial. Using the notion of infinite series, we obtain a generalization of polynomials that allows us to include terms of arbitrarily large degree. Explicitly, we define the **power series**

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots.$$

We refer to the constant $c$ as the **center** of the power series. Observe that a polynomial is nothing more than a power series for which only finitely many of the coefficients $a_i$ are nonzero.

Convergence of a power series depends not only on its sequence $a_k$ of coefficients but also on its center $c$. For instance, the power series with $a_k = k$ and $c = 0$ converges for $x = \frac{1}{2}$ but not for $x = 1$, and the power series with $a_k = k$ and $c = \frac{1}{2}$ converges for $x = 0$ but not for $x = \frac{1}{2}$.

**Example 65.** Prove that the power series $\displaystyle\sum_{k=0}^{\infty} k x^k$ converges for $x = \dfrac{1}{2}$ and diverges for $x = 1$.

**Hint:** Use the Ratio Test for $x = \dfrac{1}{2}$ and the Divergence Test for $x = 1$.

**Convergence of Power Series.** Given any power series $f(x) = \displaystyle\sum_{k=0}^{\infty} a_k (x-c)^k$, there exists an extended real number $R \geq 0$ called the **radius of convergence** with the property that

(a.) $f(x)$ converges for all $x$ such that $|x - c| < R$ and

(b.) $f(x)$ diverges for all $x$ such that $|x - c| > R$.

We refer to the interval $I = (c-R, c+R)$ on which $f(x)$ converges as the **interval of convergence**. Given that $R = \infty$, then $f(x)$ converges for all real numbers $x$, i.e., $I = (-\infty, \infty)$. Given that $R = 0$, then $f(x)$ diverges for all real numbers $x \neq c$ and $f(x)$ converges for $x = c$, i.e., $I = \{c\}$.

**Caution:** this theorem does not say anything about the convergence of the power series $f(x)$ when $x = c - R$ or $x = c + R$ for a finite, nonzero $R$; rather, we must test for convergence at these points. Unfortunately, the theorem does not give us a road map for finding the radius of convergence $R$, either. Often, we will employ the Ratio Test to this end, as it works quite well for power series.

**Example 66.** Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

**Hint:** Proceed by the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|a_{n+1} \cdot \frac{1}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{\qquad}{x^n} \cdot \frac{n!}{\qquad\qquad}\right| && \text{(Group like terms.)} \\
&= |x| \lim_{n\to\infty} \frac{n!}{(n+1)!} && \text{(Cancel, and pull out constants.)} \\
&= |x| \lim_{n\to\infty} \frac{n!}{\qquad\qquad} && \text{(Write } n! \text{ as a factor of } (n+1)!.\text{)} \\
&= |x| \lim_{n\to\infty} \frac{1}{n+1} && \text{(Cancel common factors.)} \\
&= 0.
\end{aligned}$$

Conclude that regardless of the value of $x$, the power series in question converges. Consequently, the radius of convergence is $R =$ __ and the interval of convergence is $I =$ ________.

**Example 67.** Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{3^n}$.

**Hint:** Observe that we can view this power series as a geometric series with common ratio

$$r = \underline{\qquad}.$$

Conclude by the Geometric Series Test that the power series converges if and only if $|r| < 1$. Consequently, the radius of convergence is $R =$ _ and the interval of convergence is $I =$ _____.

**Example 68.** Find the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} n^n x^n$.

Given real numbers $c$ and $r$ such that $|r| < 1$, we have that

$$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}.$$

By analogy, given a function $f(x)$ such that $|f(x)| < 1$, we find that

$$\sum_{n=0}^{\infty} c[f(x)]^n = \frac{c}{1-f(x)}.$$

Consequently, we have obtained a power series identity for any function of the form $\frac{c}{1-f(x)}$ for some function $f(x)$ that is valid for all real numbers $x$ such that $|f(x)| < 1$.

**Example 69.** Use the geometric series to find a power series identity for the following functions; then, state the radius and interval of convergence for each power series.

(a.) $\frac{1}{1-x}$

(b.) $\frac{1}{1+x}$

(c.) $\frac{1}{1+x^2}$

One of the most useful features of power series is that we may differentiate them term-by-term.

**Power Series Are Differentiable.** Consider the power series $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$ with radius of convergence $R > 0$.

(i.) We have that $f(x)$ is differentiable on the interval $I = (c-R, c+R)$ with derivative

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \sum_{k=0}^{\infty} a_k(x-c)^k = \sum_{k=0}^{\infty} a_k \frac{d}{dx}(x-c)^k = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1}.$$

Consequently, $f'(x)$ is a power series, and its radius of convergence is $R$.

(ii.) We have that the antiderivative of $f(x)$ on the interval $I = (c-R, c+R)$ is given by

$$F(x) + C = \int f(x)\,dx = \int \sum_{k=0}^{\infty} a_k(x-c)^k\,dx = \sum_{k=0}^{\infty} a_k \int (x-c)^k\,dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-c)^{k+1}.$$

Consequently, $F(x) + C = \int f(x)\,dx$ is a power series, and its radius of convergence is $R$.

We note that in practice, the constant $C$ can be found by used the fact that $F(c) + C = 0$.

**Example 70.** Use Example 69 to find a power series identity for the following functions; then, state the radius and interval of convergence for each power series.

(a.) $\dfrac{1}{(1-x)^2}$

(b.) $\ln|1+x|$

(c.) $\arctan x$

Our exploration into power series thus far has given us an infinite series expansion for any function of the form $g(x) = \frac{c}{1-f(x)}$ for some real number $c$ and function $f(x)$. Considering this as a geometric series, we have also found that this power series identity is valid for all real numbers $x$ such that $|f(x)| < 1$. By the fact above, this series can be (anti)differentiated for all $x$ such that $|f(x)| < 1$, hence we are able to obtain a power series expression for any (anti)derivative of $g(x) = \frac{c}{1-f(x)}$.

One immediate consequence of all of this is that we are now able to approximate (via power series) the value of previously unknown quantities. By Example 70(c.), we have that

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Using the fact that $\arctan(1) = \frac{\pi}{4}$, we have a representation of $\frac{\pi}{4}$ as the infinite series

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.$$

Consequently, we can approximate $\frac{\pi}{4}$ to any desired degree of accuracy, i.e., we can write the decimal expansion of $\frac{\pi}{4}$ that is accurate to as many decimal places as desired!

Certainly, we should view this as a generalization of *linearization* from Calculus I. Given a differentiable function $f(x)$ on an open interval $(a, b)$ and a real number $a < c < b$, we have that

$$f(x) \approx f(c) + f'(c) \cdot (x - c)$$

for $|x-c|$ sufficiently small by the limit definition of the derivative. We refer to the linear polynomial $T_1(x) = f(c) + f'(c)(x - c)$ as the linearization of $f(x)$ at $x = c$. Considering that $T_1(c) = f(c)$ and $T_1'(c) = f'(c)$, we say that $T_1(x)$ is a *first-order* (or linear) approximation of $f(x)$ at $x = c$. Observe that for the power series $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$, we have that

$$\begin{aligned} f(c) &= a_0 = 0! \cdot a_0, \\ f'(c) &= a_1 = 1! \cdot a_1, \\ f''(c) &= 2a_2 = 2! \cdot a_2, \\ f'''(c) &= 6a_3 = 3! \cdot a_3, \end{aligned}$$

and in general, we have that $f^{(n)}(c) = n! \cdot a_n$. Consequently, we may obtain an *nth-order* approximation of $f(x)$ at $x = c$ by generalizing degree 1 polynomial $T_1(x)$ to the degree $n$ polynomial

$$T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

**Example 71.** Prove that $T_n(x)$ is an $n$th-order approximation of $f(x)$ at $x = c$.

**Hint:** By definition, we must show that $T_n(c) = f(c)$, $T_n'(c) = f'(c)$, $T_n''(c) = f''(c)$, and in general, the $k$th derivative of $T_n(x)$ evaluated at $x = c$ is equal to $k$th derivative of $f(x)$ evaluated at $x = c$ for all integers $0 \leq k \leq n$. Explicitly, we must show that

$$\frac{d^k}{dx^k} T_n(x)\bigg|_{x=c} = \frac{d^k}{dx^k} f(x)\bigg|_{x=c} \text{ for all integers } 0 \leq k \leq n.$$

We refer to the polynomial $T_n(x)$ of degree $n$ as the **$n$th Taylor polynomial** of $f(x)$ centered at $x = c$. Crucially, we observe that the $n$th Taylor polynomial of $f(x)$ centered at $x = c$ satisfies

$$T_n(x) = T_{n-1}(x) + \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

**Uniqueness of Taylor Polynomials.** The $n$th Taylor polynomial of $f(x)$ centered at $x = c$ is the unique polynomial of degree (at most) $n$ that approximates $f(x)$ to order $n$ at $x = c$.

We shall soon see that many of the familiar functions from Calculus I can be written as power series: $e^x$, $\sin x$, and $\cos x$ all have power series expansions that converge for all real numbers $x$! Our aim is to generalize the $n$th Taylor polynomial $T_n(x)$ to a power series, but before we get there, we must first master the technique of writing the $n$th derivative of a function in a closed form.

**Example 72.** Give a closed form for the sequence $a_n = f^{(n)}(x)$ of derivatives of $f(x) = e^x$. Use this to find the $n$th Taylor polynomial $T_n(x)$ of $e^x$ centered at $x = 0$.

**Example 73.** Give a closed form for the sequence $a_n = f^{(n)}(x)$ of derivatives of $f(x) = \cos x$. Use this to find the $n$th Taylor polynomial $T_n(x)$ of $\cos x$ centered at $x = 0$.

Of course, we have said all along that our aim has been to use power series to *approximate*, and as with any approximation, there is some amount of error involved.

**Error Bound Theorem.** Consider a function $f(x)$ such that $f^{(n+1)}(x)$ exists and is continuous. Let $K$ be any real number such that $|f^{(n+1)}(u)| \leq K$ for all $u$ between $c$ and $x$. We have that

$$|f(x) - T_n(x)| \leq K \cdot \frac{|x-c|^{n+1}}{(n+1)!},$$

where $T_n(x)$ is the $n$th Taylor polynomial of $f(x)$ centered at $x = c$.

**Example 74.** Use the Error Bound Theorem to find the maximum error in approximating $e^2$ with $f(x) = e^x$ and the fourth Taylor polynomial $T_4(x)$ centered at $x = 0$.

**Example 75.** Use the Error Bound Theorem to find an integer $n \geq 0$ such that

$$|\cos(1) - T_n(1)| \leq \frac{1}{1000}.$$

Consider a function $f(x)$ such that $f^{(n)}(x)$ exists and is continuous for all integers $n \geq 0$ and all real numbers $x$ in some interval $I$. Let $K$ be a real number such that $|f^{(n)}(x)| \leq K$ for all integers $n \geq 0$ and all real numbers $x$ in $I$. By Example 36 and the Error Bound Theorem, we have that

$$\lim_{n\to\infty} |f(x) - T_n(x)| \leq \lim_{n\to\infty} K \cdot \frac{|x-c|^{n+1}}{(n+1)!} = K \cdot \lim_{n\to\infty} \frac{|x-c|^{n+1}}{(n+1)!} = 0.$$

Consequently, as the degree $n$ grows arbitrarily large, the error in approximating $f(x)$ via its $n$th Taylor polynomial centered at $x = c$ converges to 0. Further, observe that

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = T(x)$$

is a power series. We refer to $T(x)$ as the **Taylor series** of $f(x)$ centered at $x = c$. Conveniently, if a function $f(x)$ is represented by a power series centered at $x = c$ in an interval $(c - R, c + R)$ for some real number $R > 0$, then that power series is the Taylor series of $f(x)$ centered at $x = c$.

**Uniqueness of Taylor Series.** The Taylor series of $f(x)$ centered at $x = c$ is the unique power series representation (if it exists) of a function in an interval $(c - R, c + R)$ with $R > 0$.

**Caution:** this theorem does not guarantee that a function has a power series representation; rather, it says that if $f(x)$ has a power series expansion at $x = c$, then it must be the Taylor series.

We refer to the Taylor series expansion of $f(x)$ centered at $x = 0$ as the **Maclaurin series** of $f(x)$.

**Example 76.** Use Example 72 to find the Maclaurin series for $f(x) = e^x$.

**Example 77.** Use Example 73 to find the Maclaurin series for $f(x) = \cos x$.

**Example 78.** Use Example 77 to find the Maclaurin series for $f(x) = \sin x$.

Our next fact takes care (in part) of the proviso under "Uniqueness of Taylor Series."

**Convergence of Taylor Series.** Given real numbers $c$ and $R > 0$, consider the open interval $I = (c - R, c + R)$. Consider a function $f(x)$ such that $f^{(n)}(x)$ exists and is continuous for all integers $n \geq 0$ and all $x$ in $I$. If there exists a real number $K$ such that $|f^{(n)}(x)| \leq K$ for all integers $n \geq 0$ and all $x$ in $I$, then the Taylor series of $f(x)$ centered at $x = c$ converges to $f(x)$, i.e.,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-c)^n.$$

**Example 79.** Use Examples 76, 77, and 78 in addition to the above fact to find the power series expansions of $e^x$, $\cos x$, and $\sin x$. Determine the radius and interval of convergence for each of these.

Once we know the Taylor series expansion of some function $f(x)$ centered at $x = c$, it is quite easy to find the Taylor series expansion of $x^i f(x)$ or $\frac{f(x)}{x^i}$ for some integers $i \geq 1$ and the composite function $g \circ f(x)$ for some functions $g(x)$ because Taylor series play well with function composition; however, it is possible to change the center of a Taylor series when performing these operations.

**Example 80.** Use Example 79 to find the Taylor series of the following; then, state their centers.

(a.) $f(x) = x^3 \cos x$

(b.) $g(x) = e^{1-x^2}$

(c.) $h(x) = e^{x-4}$

(d.) $k(x) = \dfrac{x - \sin x}{x}$

One of the most ingenious uses of power series is to compute limits and to find power series representations for the antiderivatives of certain functions that lack elementary antiderivatives.

**Example 81.** Verify that L'Hôpital's Rule can be used to compute the limit

$$\lim_{x \to 0} \frac{x - \sin x}{x^3 \cos x};$$

then, explain the difficulty in doing so. Ultimately, compute the limit using power series.

**Example 82.** Explain the difficulty in trying to find the antiderivative of $\sin(x^2)$; then, compute the power series expansion of the antiderivative $\sin(x^2)$, and state its radius of convergence.

**Example 83.** Explain the difficulty in trying to find the antiderivative of $e^{1-x^2}$; then, compute the power series expansion of the antiderivative $e^{1-x^2}$, and state its radius of convergence.

# The Area Between Curves

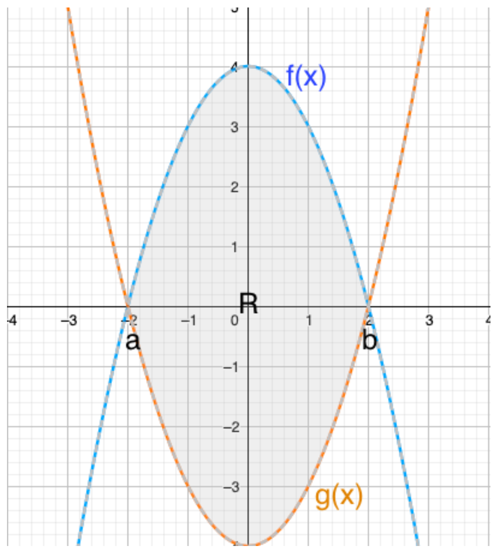
Our introduction to the notion of integration already gave us an interpretation of the definite integral $\int_a^b f(x)\,dx$ as the (signed) area between the curve $f(x)$ and the $x$-axis. Consequently, there are myriad benefits of using a definite integral to capture information about real-life observations: if we can relate a function $f(x)$ to its antiderivative $F(x)$ by observing that $f(x)$ is the rate of change of $F(x)$, then the definite integral $\int_a^b f(x)\,dx = F(b) - F(a)$ measures the total change of the function $F(x)$ from a point $x = a$ to a point $x = b$. For instance, if $f(t)$ is the velocity of a ball observed from time $t = a$ to $t = b$, then the definite integral $\int_a^b f(t)\,dt = F(b) - F(a)$ is the total distance travelled by the ball during the time frame in which we observed it.

Crucially, we can view the $x$-axis as the curve $y = g(x) = 0$, hence if our function $f(x)$ satisfies $f(x) \geq g(x) = 0$ for all $a \leq x \leq b$, then the definite integral $\int_a^b f(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$ measures the area between the curve $f(x)$ and $g(x)$. Generalizing this notion gives us a way to measure the area between any two curves $f(x)$ and $g(x)$ satisfying $f(x) \geq g(x)$ for all $a \leq x \leq b$.

**The Area Between Two Curves.** Consider the region $\mathcal{R}$ cut out by the functions $f(x)$ and $g(x)$ that satisfy $f(x) \geq g(x)$ for all $a \leq x \leq b$. We have that

$$\operatorname{area}(\mathcal{R}) = \int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Pictorially, we have the following setup.




We outline the method to find the area bounded by the curves $f_1(x)$, $f_2(x)$, $x = a$, and $x = b$.

(1.) Choose several appropriate $x$-values for the function $f_1(x)$, i.e., values of $x$ for which you know (or can easily approximate and accurately plot) the point $(x, f_1(x))$. Use the general rule that if $f_1$ is a polynomial of degree $n$, it is best to choose $n + 1$ different $x$-values to plot $f_1(x)$.

(2.) Plot the corresponding points $(x, f_1(x))$, and use these to sketch the graph of $f_1(x)$.

(3.) Repeat points (1.) and (2.) for $f_2(x)$.

(4.) Label the top function as $f(x)$ and the bottom function as $g(x)$ based on the graphs.

(5.) Use the above fact to compute the area of $\mathcal{R}$.

**Example 84.** Compute the area of the region $\mathcal{R}$ cut out by the curves $f(x) = -x^2 + 4$ and $g(x) = x^2 - 4$ for all $-2 \leq x \leq 2$.

**Example 85.** Compute the area of the region $\mathcal{R}$ cut out by the curves $f(x) = 2x + 1$ and $g(x) = 2x - 4$ for all $-1 \leq x \leq 2$. Explain how one can use geometry to verify that this area is correct. Last, discuss what would happen if we were not given values of $a$ and $b$ such that $a \leq x \leq b$.

**Example 86.** Compute the area of the region $\mathcal{R}$ cut out by the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$.

**Hint:** First, find real numbers $a$ and $b$ such that $f(x) \geq g(x)$ or $g(x) \geq f(x)$ for all $a \leq x \leq b$. One can accomplish this by checking when $f(x) = g(x)$. Once this is finished, check by inspection whether $f(x) \geq g(x)$ or $g(x) \geq f(x)$ for all $a \leq x \leq b$.

Consider a region $\mathcal{R}$ in the Cartesian plane $\mathbb{R}^2$. Quite generally, we say the region $\mathcal{R}$ is **vertically simple** if there exist functions $f_1(x)$ and $f_2(x)$ and real numbers $a$ and $b$ such that $f_1(x) \leq y \leq f_2(x)$ for all $a \leq x \leq b$. Graphically, the function $f_1(x)$ is the "bottom" function, and the function $f_2(x)$ is the "top" function, so colloquially, we refer to $f_1(x)$ as $y_{\text{bottom}}$ and $f_2(x)$ as $y_{\text{top}}$.

**The Area of a Vertically Simple Region.** Given a vertically simple region $\mathcal{R}$ cut out by the functions $y_{\text{top}} = f(x)$ and $y_{\text{bottom}} = g(x)$ for all $a \le x \le b$, we have that

$$\text{area}(\mathcal{R}) = \int_a^b (y_{\text{top}} - y_{\text{bottom}})\, dx = \int_a^b [f(x) - g(x)]\, dx.$$

Our regions have been thus far vertically simple, hence we have been able to compute their areas using the above formula. Unfortunately, there exist regions that are not vertically simple.

**Example 87.** Prove that the region $\mathcal{R}$ cut out by the curves $y = x$, $y = -x$, and $y = -2$ is not vertically simple; then, write the region $\mathcal{R}$ as the union of two vertically simple regions $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, and find the area of $\mathcal{R}$ by using the fact that $\text{area}(\mathcal{R}_1 \cup \mathcal{R}_2) = \text{area}(\mathcal{R}_1) + \text{area}(\mathcal{R}_2)$. Check that your final answer is correct using elementary geometry.

**Hint:** On the contrary, if $\mathcal{R}$ were vertically simple, then there would exist well-defined curves $y_{\text{top}}$ and $y_{\text{bottom}}$ for all $a \le x \le b$. Prove that this is not the case by exhibiting two intervals $a \le x \le$c and $c \le x \le b$ such that the $y_{\text{top}}$ curves are not the same on both intervals. Use the fact that on each interval $a \le x \le c$ and $c \le x \le b$, there are well-defined curves $y_{\text{top}}$ and $y_{\text{bottom}}$ to show that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ for some vertically simple regions $\mathcal{R}_1$ with $a \le x \le c$ and $\mathcal{R}_2$ with $c \le x \le b$.

We have just exhibited a region $\mathcal{R}$ that is not vertically simple; however, if we were to change the names of $x$ and $y$, then we would find that our region is simple. Explicitly, we say that the region $\mathcal{R}$ is **horizontally simple** if there exist functions $g_1(y)$ and $g_2(y)$ and real numbers $c$ and $d$ such that $g_1(y) \le x \le g_2(y)$ for all $c \le y \le d$. Graphically, the function $g_1(y)$ is the "left" function, and the function $g_2(y)$ is the "right" function, so colloquially, we refer to $g_1(y)$ as $x_{\text{left}}$ and $g_2(y)$ as $x_{\text{right}}$.

**The Area of a Horizontally Simple Region.** Given a horizontally simple region $\mathcal{R}$ cut out by the functions $x_{\text{right}} = f(y)$ and $x_{\text{left}} = g(y)$ for all $c \le y \le d$, we have that

$$\text{area}(\mathcal{R}) = \int_c^d (x_{\text{right}} - x_{\text{left}})\, dy = \int_c^d [f(y) - g(y)]\, dy.$$

**Example 88.** Prove that the region $\mathcal{R}$ of Example 87 is horizontally simple by exhibiting well-defined curves $x_{\text{left}}$ and $x_{\text{right}}$ for all $c \le y \le d$; then, compute the area of $\mathcal{R}$.

On the other hand, it is possible for function to be both vertically and horizontally simple.

**Example 89.** Describe the region in Example 86 as horizontally simple. List any observations you have about your description of the region; then, compute its area.

**Hint:** Exhibit well-defined curves $x_{\text{right}} = g_2(y)$ and $x_{\text{left}} = g_1(y)$ for all $c \le y \le d$.

Likewise, it is possible for a region to be neither vertically nor horizontally simple.

**Example 90.** Prove that the region $\mathcal{R}$ enclosed by the curves $y = x - 2$, $y = 2 - x$, $y = -x + 2$, and $y = -x - 2$ is neither vertically nor horizontally simple.

**Hint:** Use the symmetry of $\mathcal{R}$ to argue that it suffices to check only that $\mathcal{R}$ is not vertically (or horizontally) simple; then, proceed to show that $\mathcal{R}$ is not vertically (or horizontally) simple.

Our above exposition completely determines how to compute the area of a region as soon as we can identify it as vertically or horizontally simple; however, there remains some nuance to these types of problems. Our previous example establishes the existence of regions that are neither vertically nor horizontally simple, so the question remains as to how we deal with these. One strategy is to break up such a region into subregions that are either vertically or horizontally simple. (Later, in Calculus III, we will learn the change of variables method that will make this issue more manageable.)

On the other hand, it is also completely possible that we are handed a region that is both vertically and horizontally simple, and the description of the region as vertically simple makes the integral very difficult to compute. Our best bet in this case is to check the description of the region as horizontally simple and hope that the integrand works out to be nicer in this lens.

**Example 91.** Compute $-\int_0^1 \ln x \, dx$ by viewing it as the area of some region $\mathcal{R}$.

**Hint:** Considering that $\ln(x) \leq 0$ for all $0 < x \leq 1$, it follows that $-\int_0^1 \ln x \, dx$ is the area of the region $\mathcal{R}$ bounded by the curves $y = 0$, $y = \ln x$, $x = 0$, and $x = 1$. Can we describe $\mathcal{R}$ as horizontally simple? If so, then we would have that $-\int_0^1 \ln x \, dx = \int_c^d (x_{\text{right}} - x_{\text{left}}) \, dy$.

# Volume, Density, and Average Value

By the previous section, given a region $\mathcal{R}$ bounded by several curves, we can find the area of $\mathcal{R}$ by viewing the region as the union of some vertically (or horizontally) simple subregions and summing up the respective areas of each region. Consequently, we might suspect that a similar approach would work to compute the volume of a three-dimensional solid.

Explicitly, consider a three-dimensional solid $\mathcal{S}$. By taking $n$ vertical slices of $\mathcal{S}$ of equal width $\Delta x$ and examining the resulting *cross sections* of $\mathcal{S}$, we can approximate the volume of $\mathcal{S}$ by

$$\text{volume}(\mathcal{S}) \approx \sum_{k=1}^{n} \text{area}(\mathcal{S}_k)\Delta x,$$

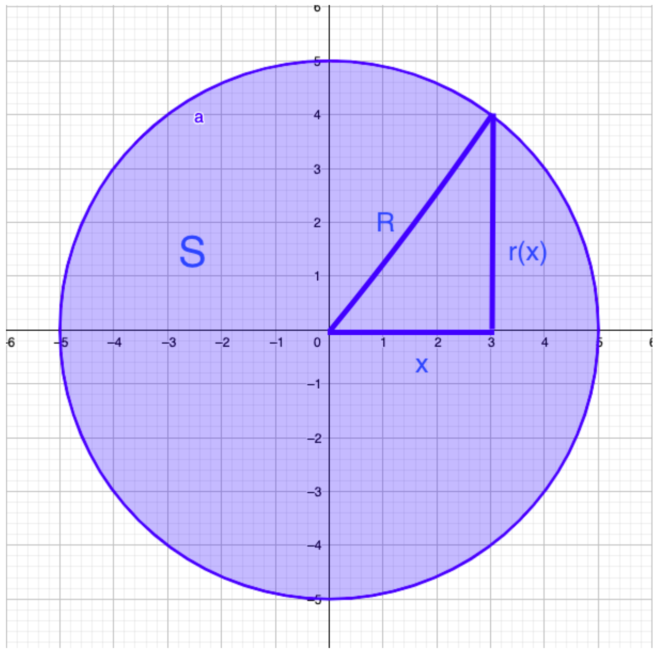
where $\text{area}(\mathcal{S}_k)$ is the **cross-sectional area** of the $k$th slice $\mathcal{S}_k$ of $\mathcal{S}$. For instance, if we have a sphere $\mathcal{S}$ of radius $R = 1$, then we can take $n$ vertical slices of the sphere. Each slice $\mathcal{S}_k$ is a circle of radius $r_k(x)$, hence we have that $\text{area}(\mathcal{S}_k) = \pi[r_k(x)]^2$ and $\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}$. We know from precalculus that the volume of the sphere of radius $R$ is $\frac{4}{3}\pi R^3$, and we will soon demonstrate this. By viewing the above sum as a Riemann sum and taking the limit as $n$ approaches $\infty$, we have that

$$\text{volume}(\mathcal{S}) = \int_a^b \text{area}(\mathcal{S}) \, dx,$$

where $\text{area}(\mathcal{S})$ is the cross-sectional area of any slice of $\mathcal{S}$.

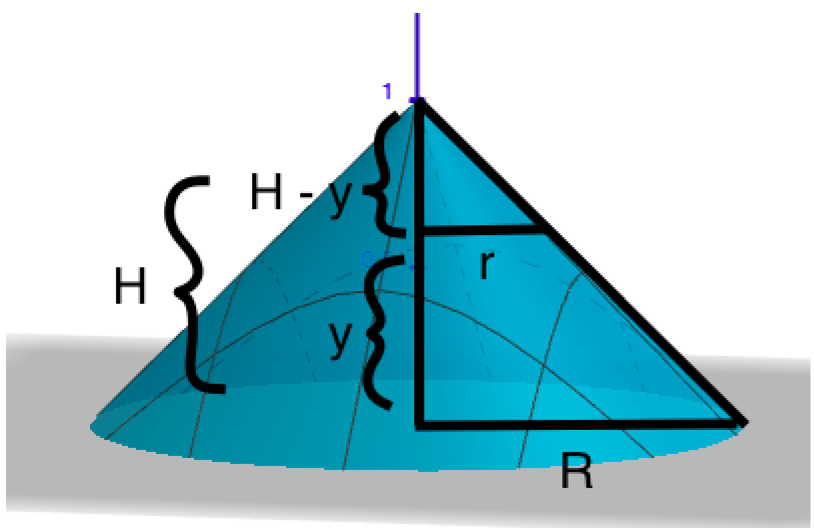
**Example 92.** Prove that the volume of a sphere of radius $R > 0$ is $\frac{4}{3}\pi R^3$.

**Hint:** By the paragraph above, we can use the formula for the volume by cross-sectional area. Our cross sections may be taken vertically (i.e., perpendicular to the $x$-axis). Each of these is a circle of radius $r(x)$. Prove that $r(x) = \sqrt{R^2 - x^2}$ by using the Pythagorean Theorem on the diagram below; then, use the fact that area$(\mathcal{S}) = \pi[r(x)]^2$ and the formula above to finish.




**Example 93.** Prove that the volume of a right-circular cone of radius $R$ and height $H$ is $\frac{1}{3}\pi R^2 H$.

**Hint:** By the paragraph above, we can use the formula for the volume by cross-sectional area. Observe that the horizontal cross sections of a right-circular cone $\mathcal{C}$ of radius $R$ and height $H$ are circles of radius $r(y)$. Prove that $r(y) = \frac{R}{H}(H - y)$ by using similar triangles on the diagram below; then, use the fact that area$(\mathcal{C}) = \pi[r(y)]^2$ and the formula above to finish.




Recall from physics that the mass $m$ of an object of length $\ell$ and constant lineal density $\rho$ is given by $m = \rho \cdot \ell$. Of course, not all objects have constant density: a pencil might be more dense toward its point and less dense toward its eraser for improved comfort when writing. Consequently, we can describe its lineal density as a function $\rho(x)$, where $x$ measures the distance from the tip of the pencil to its eraser. One natural question arises: "What is the mass of this pencil?"

Like usual, if we break the pencil $\mathcal{P}$ of length $b-a$ up into $n$ vertical slices of equal width $\Delta x$, then we can approximate the mass of $\mathcal{P}$ by the Riemann sum

$$\text{mass}(\mathcal{P}) \approx \sum_{k=1}^{n} \rho(P_k)\Delta x,$$

where $\rho(P_k)$ is the lineal density of a point $P_k$ in the $k$th slice of $\mathcal{P}$. By taking the limit as the number of points $n$ tends to $\infty$, we reduce our error to zero, and we obtain

$$\text{mass}(\mathcal{P}) = \int_a^b \rho(x)\,dx.$$

**Example 94.** Compute the total mass of a rod of length 1 unit and lineal density $\rho(x) = xe^{x^2}$.

Given a list of $n$ values $a_1, \dots, a_n$, recall that the *average* of these values is given by

$$\frac{a_1 + \cdots + a_n}{n} = \frac{1}{n}\sum_{k=1}^{n} a_k.$$

Consequently, we may use this approach if we wish to approximate the average value of a function $f(x)$ that is integrable on a closed interval $[a, b]$. Explicitly, we may choose $n$ values $f(x_1), \dots, f(x_n)$ for some equally-spaced real numbers $a = x_1 \leq \cdots \leq x_n = b$. Using the fact that $\Delta x = \frac{b-a}{n}$ is the distance between any two consecutive $x$-values, our above displayed equation gives that

$$\text{average value of } f(x) \text{ on } [a,b] \approx \frac{1}{n}\sum_{k=1}^{n} f(x_k) = \frac{1}{b-a}\cdot\frac{b-a}{n}\sum_{k=1}^{n} f(x_k) = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\Delta x.$$

By recognizing this as a Riemann sum as taking the limit as $n \to \infty$, we find that

$$\text{average value of } f(x) \text{ on } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

**Example 95.** Compute the average value of the function $f(x) = x^{-1}$ on the interval $\left[\frac{1}{e}, 1\right]$.

One of the most important applications of the average value of a function is the following.

**The Mean Value Theorem for Integrals.** Consider a function $f(x)$ that is continuous (and therefore integrable) on a closed interval $[a, b]$. There exists a real number $a \leq c \leq b$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

**Example 96.** Consider a car travelling with a velocity of $v(t)$ units per minute. Prove that if the car enters a 325 unit-long tunnel at $t = 0$ minutes and exits at $t = 4$ minutes and the speed limit in the tunnel is 80 units per minute, then the car broke the speed limit at some point in the tunnel.