MATH 125: Useful Expressions

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Chain Rule

- "We need to use the Chain Rule to differentiate functions that are well-guarded fortresses."

  Think of a function $f(x)$ as a fortress protecting the thing you want most in life x, e.g., a silverface Fender Super Reverb guitar amplifier. We use the Chain Rule to systematically disarm each of the defenses the fortress $f(x)$ presents until at last we obtain x.

  Bear in mind that some functions are more "well-guarded" than others. For instance, if

  $$f(x) = e^{\sin(x^2)\cos(x^3)} \text{ and } g(x) = \ln(3x + 2),$$

  the function $f(x)$ is much more complicated (or "well-guarded") than $g(x)$. In order to differentiate $f(x)$, we must first identify each line of defense of $f(x)$.

  1.) Outside $f(x)$, there is a mote with an alligator in it that represents e^{exponent}. By taking the derivative of e^{exponent}, we are able to pass by the alligator, leaving it behind us.

  2.) Once past the alligator, we are confronted by a knight that represents $\sin(x^2)\cos(x^3)$. By taking the derivative of $\sin(x^2)\cos(x^3)$, we are able to pass the knight, leaving it behind.

  3.) Both $\sin(x^2)$ and $\cos(x^3)$ are armed with rune armor and dragon scimitars, so we must disarm both of them to pass onto the inside terms of x^2 and x^3, respectively.

  All together, we arrive at the following.

  $$f'(x) = \underbrace{e^{\sin(x^2)\cos(x^3)}}_{\frac{d}{dx}(\text{alligator})} \cdot \left(\underbrace{\sin(x^2) \cdot -\sin(x^3) \cdot \overbrace{3x^2}^{\frac{d}{dx}(\text{dragon scimitar})} + \cos(x^3) \cdot \cos(x^2) \cdot \overbrace{2x}^{\frac{d}{dx}(\text{rune armor})}}_{\frac{d}{dx}(\text{knight})} \right)$$

  On the other hand, $g(x)$ only offers the protection of an alligator (the natural logarithm) and a staircase (the inner function $3x + 2$), so we find that the following holds.

  $$g'(x) = \underbrace{\frac{1}{3x+2}}_{\frac{d}{dx}(\text{alligator})} \cdot \underbrace{3}_{\frac{d}{dx}(\text{staircase})}$$

- “Don’t try to fight the alligator and the armed guard at the same time.”

  It is a common mistake for students to incorrectly implement the Chain Rule as

  $$\frac{d}{dx} f(g(x)) = f'(g'(x)).$$

  But this is the equivalent of fighting both the alligator and the armed guard at the same time. It is way more difficult — and mathematically, it is incorrect.

Continuity and Differentiability

- “We don’t want Mine Cart Madness or Mine Cart Carnage.”

  A function is continuous if and only if its graph can be sketched without lifting one’s pencil. Particularly, there are no holes, jumps, or vertical asymptotes.

  A function is differentiable if and only if its graph can be sketched without lifting one’s pencil and there are no cusps, sharp corners, or vertical tangents in the graph.

- “Continuous functions play nice with limits.”

  If $f(x)$ is continuous at $x = a$, then the limit is evaluated by direct substitution, i.e.,

  $$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right) = f(a).$$

  Because of this, we say that continuous functions “play nice” (or commute) with limits.

Factoring Polynomials

- “Difference of squares? Conjugate pairs!”

  The difference of two perfect squares a^2 and b^2 always factors as the product of the pair of conjugates $a + b$ and $a - b$. Put another way, we always have that $a^2 - b^2 = (a + b)(a - b)$.

Limits

- “The limit is a promise.”

  In the language of mathematics, we write,

  $$\lim_{x \to a} f(x) = L.$$

  In English, we say, “No matter how close you want the function $f(x)$ to be to the real number L, I promise I can find a real number x near a — but not equal to a — so that happens.”

- “Plug it in.”

  The first (naïve) thing to do when evaluating a limit is simply to plug in the x-value. Best case scenario, the limit is of determinate form (i.e., it is $\pm\infty$ or a real number).

- "Don't hesitate; conjugate."

  If you encounter one of the following situations, multiply and divide by the conjugate.

  - the difference of square roots whose limits are both infinity, e.g.,

$$\lim_{x \to \infty} \left(\sqrt{49x^2 - x + 7} - \sqrt{25x^2 + 3x} \right)$$

  - the difference of a square root and a polynomial whose limits are both infinity, e.g.,

$$\lim_{x \to \infty} \left(\sqrt{81x^4 - 9x^2 + 1} - 3x^2 + 17 \right)$$

- "Don't hesitate; least common 'denominate.'"

  If you encounter one of the following situations, find the least common denominator; then, multiply and divide each fraction by what is *missing from* the denominator.

  - the difference of rational functions whose limits are both infinity, e.g.,

$$\lim_{h \to 0} \left(\frac{1}{h(9+h)} - \frac{1}{9h} \right)$$

Piecewise Functions

- "The absolute value function is a conveyor belt."

  The absolute value function $|x|$ takes an input x and first checks if x is non-negative. If it is, then $x \geq 0$, and the absolute value function places x back on the conveyor belt. If x is negative, then $x < 0$, and the absolute value function places a minus sign on x before it puts it on the conveyor belt. In the language of mathematics, we write the following.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Product Rule

- "The derivative of the product of two differentiable functions is the first times the derivative of the second plus the second times the derivative of the first."

  Given any differentiable functions $f(x)$ and $g(x)$, we have that

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Quotient Rule

- "The derivative of the quotient of two differentiable functions is low-dee-high minus high-dee-low all over low-squared."

  Given any differentiable functions $f(x)$ and $g(x)$, we have that

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Trigonometric Limits

- "Be sure to pair each function with its condiment."

  There are two fundamental trigonometric limits that govern most other trigonometric limits:

  (a.) $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ and

  (b.) $\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$.

  Put another way, in the first trigonometric limit, the "condiment" of $\sin(x)$ is x, and in the second trigonometric limit, the "condiment" of $1-\cos(x)$ is x.

- "Don't put out the sandwiches until they all have condiments."

  Computing trigonometric limits is like making a bunch of trigonometric function sandwiches and trying to pair each one with its condiment. For instance, in the following limit

  $$\lim_{x\to 0} \frac{1-\cos(7x)}{\sin(3x)} = \lim_{x\to 0} \left(\frac{1-\cos(7x)}{1} \cdot \frac{1}{\sin(3x)} \right),$$

  the $\sin(3x)$ in the denominator is missing its condiment $3x$. We pair it with its condiment $3x$ by multiplying and dividing by $3x$ to obtain

  $$\lim_{x\to 0} \frac{1-\cos(7x)}{\sin(3x)} = \lim_{x\to 0} \left(\frac{1-\cos(7x)}{3x} \cdot \frac{3x}{\sin(3x)} \right).$$

  But now, the $1-\cos(7x)$ in the numerator needs its condiment, too! Recalling that the condiment of $1-\cos(7x)$ is $7x$, we multiply and divide by 7 to obtain

  $$\lim_{x\to 0} \frac{1-\cos(7x)}{\sin(3x)} = \lim_{x\to 0} \left(\frac{7}{3} \cdot \frac{1-\cos(7x)}{7x} \cdot \frac{3x}{\sin(3x)} \right).$$

  Both $1-\cos(7x)$ and $\sin(3x)$ have their condiments, so we can now "put out the sandwiches."

  $$\lim_{x\to 0} \frac{1-\cos(7x)}{\sin(3x)} = \lim_{x\to 0} \frac{7}{3} \cdot \lim_{x\to 0} \frac{1-\cos(7x)}{7x} \cdot \lim_{x\to 0} \frac{3x}{\sin(3x)} = \frac{7}{3} \cdot 1 \cdot 1.$$

- "There are trigonometric limits that require more toppings than just condiments."

  Occasionally, we will encounter a trigonometric limit in which some function will not have "enough condiments." We refer to these trigonometric limits as "more sophisticated sandwiches." For instance, the following limit needs more toppings than just condiments.

  $$\lim_{x\to 0} \frac{1-\cos(4x)}{\sin^2(3x)} = \lim_{x\to 0} \left(\frac{1-\cos(4x)}{1} \cdot \frac{1}{\sin(3x)} \cdot \frac{1}{\sin(3x)} \right)$$

  Essentially, the problem is that we need to give each of the four factors of $\sin(3x)$ its condiment $3x$, and we need to give $1-\cos(4x)$ its condiment $4x$, but then, we will have a factor of $\frac{4x}{(3x)^2}$, and the limit as x tends to 0 of this function does not exist. But if we add a slice of cheese by multiplying and dividing by the conjugate $1+\cos(4x)$, the sandwich can be completed.

$$\lim_{x\to 0}\frac{1-\cos(4x)}{\sin^4(3x)}\cdot\frac{1+\cos(4x)}{1+\cos(4x)}=\lim_{x\to 0}\frac{1-\cos^2(4x)}{\sin^2(3x)(1+\cos(4x))}=\lim_{x\to 0}\frac{\sin^2(4x)}{\sin^2(3x)(1+\cos(4x))}$$

We have not changed the function whose limit we are taking, but we now have the following.

$$\lim_{x\to 0}\frac{\sin^2(4x)}{\sin^2(3x)(1+\cos(4x))}=\lim_{x\to 0}\frac{\sin(4x)}{1}\cdot\frac{\sin(4x)}{1}\cdot\frac{1}{\sin(3x)}\cdot\frac{1}{\sin(3x)}\cdot\frac{1}{1+\cos(4x)}$$

By applying to each function its appropriate condiment, we obtain the following.

$$\lim_{x\to 0}\frac{\sin^2(4x)}{\sin^4(3x)(1+\cos(4x))}=\lim_{x\to 0}\frac{16x^2}{9x^2}\cdot\frac{\sin(4x)}{4x}\cdot\frac{\sin(4x)}{4x}\cdot\frac{3x}{\sin(3x)}\cdot\frac{3x}{\sin(3x)}\cdot\frac{1}{1+\cos(4x)}$$

Cancelling a factor of x^2 from the numerator and denominator and using the fact that $\lim_{x\to 0}(1+\cos(4x))=2$, we can "put out the sandwiches."

$$\lim_{x\to 0}\frac{\sin^2(4x)}{\sin^4(3x)(1+\cos(4x))}=\frac{16}{9}\cdot 1\cdot 1\cdot 1\cdot 1\cdot\frac{1}{2}$$