Classical Trigonometric Limits

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We present two classical trigonometric limits, proving one of them; the other is left to the reader.

Theorem. We have that $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$.

Proof. Given any angle θ in $(0, \frac{\pi}{2}]$, it can be shown geometrically that $\sin(\theta) \leq \theta \leq \tan(\theta)$. By dividing each term in this inequality by $\sin(\theta)$, we have that

$$1 \leq \frac{\theta}{\sin(\theta)} \leq \frac{1}{\cos(\theta)},$$

from which it follows by inverting each term that we have

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1.$$

By taking the limit of each term, we conclude (by Squeeze Theorem) that $\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1$. By a similar analysis for θ in $[-\frac{\pi}{2}, 0)$, we have that $\lim_{\theta \to 0^-} \frac{\sin(\theta)}{\theta} = 1$ so that $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$. □

Theorem. We have that $\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0$.

Proof. We leave the proof to the reader. □