Practice Algebra Qual II

August 2020

We adopt the shorthand notation $\mathbb{Z}_n \stackrel{\text{def}}{=} \mathbb{Z}/n\mathbb{Z}$.

1.) Prove that $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are the only groups of order four (up to isomorphism) without appealing to the Fundamental Theorem of Finitely Generated Abelian Groups.

2.) Given a commutative unital ring R, consider the ring of Laurent polynomials $R[t, t^{-1}]$.

   (a.) Prove that $R[t, t^{-1}] \cong R[x, y]/(xy - 1)$.

   (b.) Prove that if k is a field, then $k[t, t^{-1}]$ is a principal ideal domain.

   (c.) Prove that if k is a field, then $k[t, t^{-1}]$ is a Euclidean domain.

3.) Let p be an odd prime number.

   (a.) Prove that for any integer a such that $a \not\equiv 0 \pmod{p}$, the congruence $x^2 \equiv a \pmod{p}$ has a solution in $\mathbb{Z}$ if and only if $a^{(p-1)/2} \equiv 1 \pmod{p}$.

   (b.) Using part (a.), determine whether or not the polynomial $x^2 - 6$ is irreducible in $\mathbb{Z}_{17}[x]$.

4.) Consider a field k whose multiplicative group $k^\times$ of units is cyclic. Prove that k is finite.

5.) Consider the 3×3 matrices A and B with entries in $\mathbb{C}$ (the complex numbers). Prove that

$$\det(AB - BA) = \frac{1}{3}\,\text{trace}[(AB - BA)^3].$$

6.) Given the vector space V of 2×2 real matrices, consider the linear transformation $T_A : V \to V$ defined by $T_A(B) = ABA^{-1}$ for some invertible 2×2 real matrix A. Given that

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

find with proof the Jordan Canonical Form of the linear transformation T_A.