Practice Algebra Qual I

August 2020

1.) Consider a group G. We define the *commutator* of two elements a and b of G to be the element $[a, b] = aba^{-1}b^{-1}$ of G. Consider the *commutator subgroup* of G defined by

$$[G, G] = \left\{ \prod_{i=1}^{n} [a_i, b_i] \,\middle|\, a_i, b_i \in G \text{ and } n \geq 0 \text{ is an integer} \right\}.$$

   (a.) Prove that $[G, G]$ is normal in G.

   (b.) Prove that $G/[G, G]$ is abelian.

   (c.) Given a normal subgroup N of G such that G/N is abelian, prove that $[G, G] \subseteq N$.

2.) Consider an infinite commutative unital ring R such that $|R/I| < \infty$ for every nonzero ideal I of R. Prove that R is an integral domain.

3.) Using Eisenstein's Criterion, one can prove that the cyclotomic polynomial

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible over $\mathbb{Q}$ for every prime p. Given a root ζ of $\Phi_p(x)$, consider the field $\mathbb{Q}(\zeta)$.

   (a.) Prove that the set $\{\zeta, \zeta^2, \ldots, \zeta^{p-1}\}$ consists of distinct roots of $\Phi_p(x)$. Conclude that these are precisely all of the zeros of $\Phi_p(x)$.

   (b.) Prove that the set G of all field automorphisms $\varphi : \mathbb{Q}(\zeta) \to \mathbb{Q}(\zeta)$ is an abelian group with respect to composition. Further, prove that $|G| = p - 1$.

   (c.) Prove that the set $\{a \in \mathbb{Q}(\zeta) \,|\, \varphi(a) = a \text{ for all automorphisms } \varphi \in G\}$ is equal to $\mathbb{Q}$.

4.) Given an odd integer n, consider the subspace V of the vector space of $n \times n$ matrices over $\mathbb{R}$. Prove that if $V \setminus \{0\}$ is a sub*set* of the multiplicative group of invertible $n \times n$ real matrices $GL(n, \mathbb{R})$, then we must have that $\dim_{\mathbb{R}}(V) \leq 1$.

5.) Given a positive integer n, denote by $M(n, k)$ the ring of $n \times n$ matrices over the field k. Prove that if there is a ring isomorphism $\varphi : M(n, k) \to M(m, k)$, then we must have that $m = n$.

6.) Given a field k, consider the vector space $P_2(k)$ of polynomials in $k[x]$ of degree ≤ 2.

   (a.) Prove that the linear map $D : P_2(k) \to P_2(k)$ defined by $D(f) = f + f' + f''$ is invertible.

   (b.) Find with proof the Jordan Canonical Form for D.